

**Da se podsetimo!**

Ako je S glatka dvostrana površ na kojoj je izabrana jedna od dveju strana , određena smerom normale

$\vec{n}(\cos \alpha, \cos \beta, \cos \gamma)$  i  $z = z(x,y)$  tada je :

$$\cos \alpha = \frac{p}{\pm \sqrt{1 + p^2 + q^2}}$$

$$\cos \beta = \frac{q}{\pm \sqrt{1 + p^2 + q^2}} \quad \text{gde je: } p = \frac{\partial z}{\partial x} \quad \text{i} \quad q = \frac{\partial z}{\partial y}$$

$$\cos \gamma = \frac{-1}{\pm \sqrt{1 + p^2 + q^2}}$$

**VAŽNO**

( Da li ćemo uzeti + ili – zavisi od ugla koji normala gradi sa pozitivnim delom z-ose:

Ako je taj ugao oštar ,onda mora biti  $\cos \gamma > 0$  pa uzimamo minus ispred korena,  $\cos \gamma = \frac{-1}{-\sqrt{1 + p^2 + q^2}}$

Ako je taj ugao tup, onda je  $\cos \gamma < 0$  , pa uzimamo + ispred korena  $\cos \gamma = \frac{-1}{+\sqrt{1 + p^2 + q^2}}$  )

a  $P=P(x,y,z)$   $Q=Q(x,y,z)$  i  $R=R(x,y,z)$  tri funkcije, definisane i neprekidne na površi S, onda je

$$\iint_S P dy dz + Q dz dx + R dx dy = \iint_S (P \cos \alpha + Q \cos \beta + R \cos \gamma) ds$$

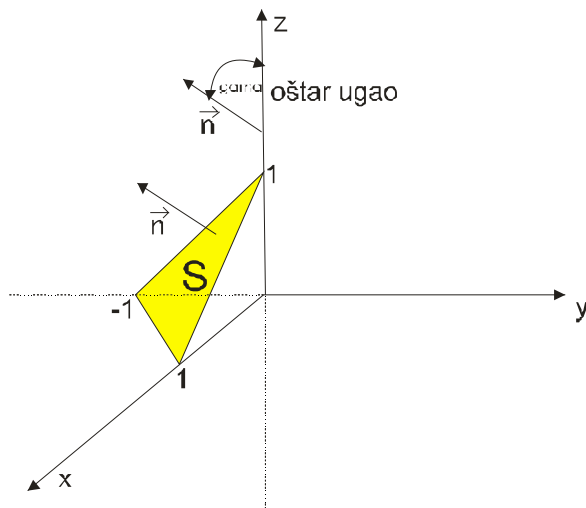
Površinski integral druge vrste **zavisi** od orijentacije krive.

**Prelaskom na drugu stranu površi menja se znak !.**

**1. Izračunati površinski integral  $I = \iint_S z dx dy + x dx dz + y dy dz$  ako je S gornji deo ravni  $x - y + z = 1$  isečen koordinatnim ravnima.**

**Rešenje:**

Nacrtajmo najpre sliku:



Iz date ravni izrazimo  $z$  pa nadjemo  $p$  i  $q$ .

$$x - y + z = 1 \rightarrow z = 1 - x + y \rightarrow p = \frac{\partial z}{\partial x} = -1 \wedge q = \frac{\partial z}{\partial y} = 1$$

Sa slike vidimo da je ugao izmedju normale na ravan i  $z$  ose (pozitivan smer gledamo) oštar, pa nam to govori da ćemo

ispred korena u imeniocu uzimati minus! (To jest, pravimo da ceo  $\cos \gamma = \frac{-1}{\pm \sqrt{1+p^2+q^2}}$  bude pozitivan)

$$\cos \gamma = \frac{-1}{-\sqrt{1+p^2+q^2}} = \frac{1}{\sqrt{1+(-1)^2+1^2}} = \frac{1}{\sqrt{3}}$$

Kad smo zaključili da je minus ispred korena, tako radimo i za ostala dva:

$$\cos \alpha = \frac{p}{\pm \sqrt{1+p^2+q^2}} = \frac{-1}{-\sqrt{1+(-1)^2+1^2}} = \frac{1}{\sqrt{3}}$$

$$\cos \beta = \frac{q}{\pm \sqrt{1+p^2+q^2}} = \frac{1}{-\sqrt{1+(-1)^2+1^2}} = -\frac{1}{\sqrt{3}}$$

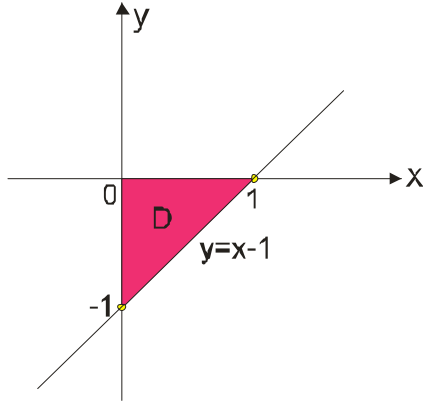
Iz datog integrala  $I = \iint_S z dx dy + x dx dz + y dy dz$  upoređujući ga sa  $\iint_S P dy dz + Q dz dx + R dx dy$

pročitamo da je  $P = y$ ,  $Q = x$  i  $R = z$ .

Sad možemo upotrebiti formulu:  $\iint_S P dydz + Q dzdx + R dx dy = \iint_S (P \cos \alpha + Q \cos \beta + R \cos \gamma) dS$

$$I = \iint_S z dx dy + x dx dz + y dy dz = \iint_S \left( y \cdot \frac{1}{\sqrt{3}} + x \cdot \left( -\frac{1}{\sqrt{3}} \right) + z \cdot \frac{1}{\sqrt{3}} \right) dS = \frac{1}{\sqrt{3}} \iint_S (y - x + z) dS$$

Vreme je da problem spustimo u ravan  $z = 0$ , nacrtamo sliku i odredimo granice.



$$z = 0 \wedge x - y + z = 1 \rightarrow x - y = 1 \rightarrow y = x - 1$$

$$D: \begin{cases} 0 \leq x \leq 1 \\ x - 1 \leq y \leq 0 \end{cases} \quad \text{I već smo videli da je: } \sqrt{1 + p^2 + q^2} = \sqrt{1 + (-1)^2 + 1^2} = \sqrt{3}$$

$$\begin{aligned} I &= \frac{1}{\sqrt{3}} \iint_S (y - x + z) dS = \frac{1}{\sqrt{3}} \iint_S (y - x + 1 - x - y) \sqrt{1 + p^2 + q^2} dx dy = \frac{1}{\sqrt{3}} \iint_S (2y - 2x + 1) dx dy = \\ &= \int_0^1 dx \int_{x-1}^0 (2y - 2x + 1) dy = \dots = -\frac{1}{6} \end{aligned}$$

## II način

Neki profesori ne vole da rade ovako, već početni integral podele na tri integrala:

$$I = \iint_S z dx dy + x dx dz + y dy dz = \underbrace{\iint_S z dx dy}_{I_1} + \underbrace{\iint_S x dx dz}_{I_2} + \underbrace{\iint_S y dy dz}_{I_3}$$

**E sad svaki integral rešavamo posebno pa ćemo sabrati rešenja!**

Za prvi integral koji radimo po  $dx dy$  izrazimo  $z$  i zamenimo, a granice odredjujemo u  $z = 0$

$$I_1 = \iint_S z dx dy = \iint_S (1 + y - x) dx dy = \int_0^1 dx \int_{x-1}^0 (1 + y - x) dy = \dots = \frac{1}{6}$$

Za drugi integral koji radimo po  $dx dz$  izrazimo  $y$  i zamenimo( ovde to ne mora, jer imamo samo  $x$  u integralu)

a granice odredjujemo u  $y = 0$

$$I_2 = \iint_S x dx dz = - \int_0^1 x dx \int_0^{1-x} dz = \dots = -\frac{1}{6}$$

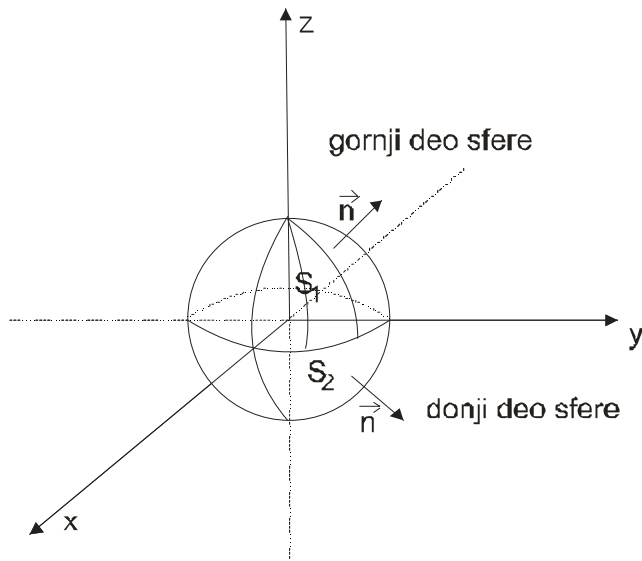
Za treći slično, pa imamo  $I_3 = \iint_S y dy dz = \int_{-1}^0 y dy \int_0^{y+1} dz = \dots = -\frac{1}{6}$

Kad ih saberemo, dobijamo rezultat  $I = \iint_S z dx dy + x dx dz + y dy dz = \underbrace{\iint_S z dx dy}_{I_1} + \underbrace{\iint_S x dx dz}_{I_2} + \underbrace{\iint_S y dy dz}_{I_3} = \frac{1}{6} - \frac{1}{6} - \frac{1}{6} = -\frac{1}{6}$

**2. Izračunati površinski integral  $I = \iint_S xyz dx dy$  ako je S spoljna strana površi  $x^2 + y^2 + z^2 = 1$ ,  $x \geq 0, y \geq 0$**

**Rešenje:**

Nacrtajmo najpre sliku da vidimo kakav ugao gradi normala....



**Šta primećujemo?**

Za gornji deo sfere  $z_1 = \sqrt{1 - x^2 - y^2}$  normala pravi oštar ugao sa pozitivnim smerom z ose a za donji deo sfere,

$z_2 = -\sqrt{1 - x^2 - y^2}$  normala pravi tup ugao sa pozitivnim smerom z ose.

Zaključujemo da moramo posebno raditi za gornji deo, posebno za donji, pa ćemo sabrati rešenja.

**ZA  $S_1$**

Kako je  $\gamma < 90^\circ \rightarrow \cos \gamma > 0 \rightarrow$  uzimamo minus ispred korena u imeniocu  $\cos \gamma = \frac{-1}{-\sqrt{1 + p^2 + q^2}} = \frac{1}{\sqrt{1 + p^2 + q^2}}$

Da nadujemo sada ove parcijalne izvode i koren na stranu ...

$$p = \frac{\partial z}{\partial x} = \frac{-x}{\sqrt{1-x^2-y^2}} \wedge q = \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{1-x^2-y^2}}$$

$$\sqrt{1+p^2+q^2} = \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{1-x^2-y^2}}\right)^2} = \sqrt{1 + \frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2}} =$$

$$= \sqrt{\frac{1-x^2-y^2+x^2+y^2}{1-x^2-y^2}} = \frac{1}{\sqrt{1-x^2-y^2}}$$

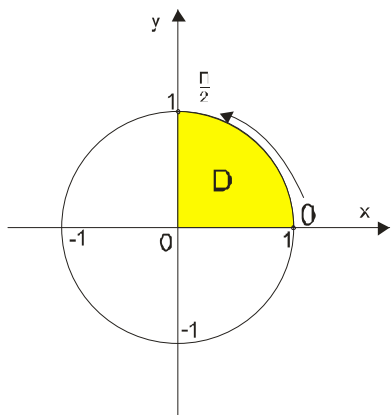
Pa je  $\cos \gamma = \frac{1}{\sqrt{1+p^2+q^2}} = \sqrt{1-x^2-y^2}$

$$I_1 = \iint_{S_1} xyz dx dy = \iint_{S_1} xyz \cos \gamma dx dy = \iint_{S_1} xy \sqrt{1-x^2-y^2} \cdot \sqrt{1-x^2-y^2} dx dy = \iint_{S_1} xy (1-x^2-y^2) dx dy$$

Spuštamo se u ravan  $z=0$ , imamo:

$$z=0 \rightarrow x^2 + y^2 = 1, x \geq 0 \wedge y \geq 0$$

Moramo opet sliku:



Nastavljamo sa zadatkom:

$$I_1 = \iint_{S_1} xy (1-x^2-y^2) dx dy = \iint_D xy (1-x^2-y^2) \sqrt{1+p^2+q^2} dx dy = \iint_D xy (1-x^2-y^2) \frac{1}{\sqrt{1-x^2-y^2}} dx dy =$$

Sad je najbolje da uzmemo:

$$\left| \begin{array}{l} x = r \cos \varphi \wedge y = r \sin \varphi \rightarrow |J| = r \\ x^2 + y^2 = 1 \rightarrow r = 1 \rightarrow D: \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \varphi \leq \frac{\pi}{2} \end{cases} \end{array} \right|$$

$$I_1 = \iint_D xy(1-x^2-y^2) \frac{1}{\sqrt{1-x^2-y^2}} dx dy = \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 r^2 \sin \varphi \cos \varphi \cdot \sqrt{1-r^2} \cdot r dr =$$

$$= \int_0^{\frac{\pi}{2}} \sin \varphi \cos \varphi d\varphi \int_0^1 r^3 \cdot \sqrt{1-r^2} dr =$$

Integral sa uglom spakujemo  $\sin \varphi \cos \varphi = \frac{1}{2} \sin 2\varphi$  a u ovom sa  $r$  uzmemo smenu  $1-r^2 = t^2$  i dobijamo:  $I_1 = \frac{1}{15}$

**Sad radimo za donji deo sfere.**

**ZA**  $S_2$  ( ovde je  $z_2 = -\sqrt{1-x^2-y^2}$  )

Kako je  $\gamma > 90^\circ \rightarrow \cos \gamma < 0 \rightarrow$  uzimamo plus ispred korena u imeniocu  $\cos \gamma = \frac{-1}{+\sqrt{1+p^2+q^2}} = -\frac{1}{\sqrt{1+p^2+q^2}}$

$$p = \frac{\partial z}{\partial x} = \frac{x}{\sqrt{1-x^2-y^2}} \wedge q = \frac{\partial z}{\partial y} = \frac{y}{\sqrt{1-x^2-y^2}}$$

$$\sqrt{1+p^2+q^2} = \sqrt{1 + \left( \frac{x}{\sqrt{1-x^2-y^2}} \right)^2 + \left( \frac{y}{\sqrt{1-x^2-y^2}} \right)^2} = \sqrt{1 + \frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2}} =$$

$$= \sqrt{\frac{1-x^2-y^2+x^2+y^2}{1-x^2-y^2}} = \frac{1}{\sqrt{1-x^2-y^2}}$$

$$\text{Pa je } \cos \gamma = -\frac{1}{\sqrt{1+p^2+q^2}} = -\sqrt{1-x^2-y^2}$$

$$I_2 = \iint_{S_2} xyz dx dy = \iint_{S_2} xyz \cos \gamma dx dy = \iint_{S_2} xy \left( -\sqrt{1-x^2-y^2} \right) \cdot \left( -\sqrt{1-x^2-y^2} \right) dx dy = \iint_{S_2} xy(1-x^2-y^2) dx dy = I_1$$

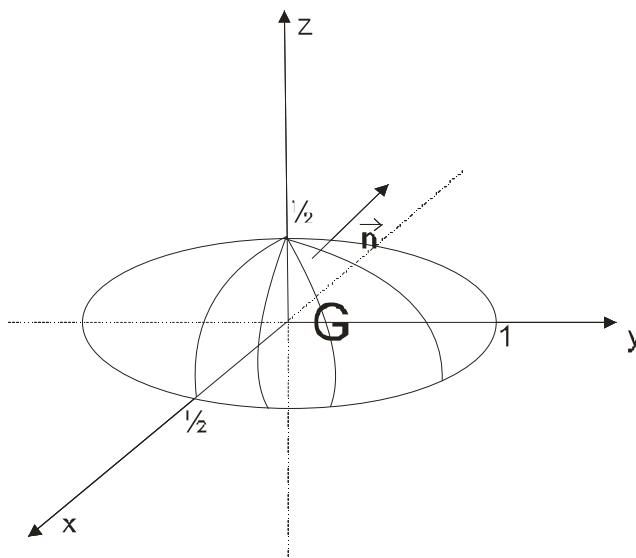
Dakle, vrednost ovog dela je takodje  $I_2 = I_1 = \frac{1}{15}$  pa je konačno:  $I = I_2 + I_1 = \frac{2}{15}$

**3. Izračunati integral  $\iint_G 2dxdy + ydzdx - x^2zdydz$  gde je G spoljna strana elipsoida  $4x^2 + y^2 + 4z^2 = 1$  koji pripada prvom oktantu.**

**Rešenje:**

Kao i uvek prvo da nacrtamo sliku i odredimo smer normale:

$$4x^2 + y^2 + 4z^2 = 1 \rightarrow \frac{x^2}{\frac{1}{4}} + \frac{y^2}{1} + \frac{z^2}{\frac{1}{4}} = 1 \rightarrow a = \frac{1}{2}, b = 1, c = \frac{1}{2}$$



Ovaj zadatak ćemo pokušati da vam objasnimo na onaj drugi način, to jest zadati integral ćemo raditi kao tri zasebna integrala pa ćemo sabrati rešenja.

$$\iint_G 2dxdy + ydzdx - x^2zdydz = \iint_G 2dxdy + \iint_G ydzdx + \iint_G -x^2zdydz = I_1 + I_2 + I_3$$

$$I_1 = \iint_G 2dxdy$$

Ovde imamo  $\frac{x^2}{\frac{1}{4}} + \frac{y^2}{1} + \frac{z^2}{\frac{1}{4}} = 1 \wedge z = 0 \rightarrow \frac{x^2}{\frac{1}{4}} + \frac{y^2}{1} = 1$

$$I_1 = \iint_{G^+} 2dxdy = 2 \iint_D dxdy = 2 \cdot m(D) = 2 \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot 1 \cdot \pi = \frac{\pi}{4}$$

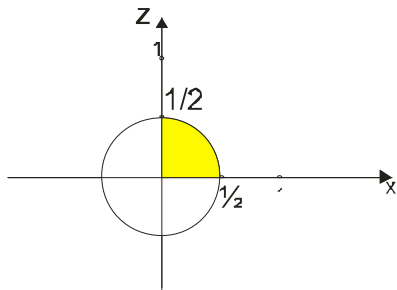
Samo da vas podsetimo da se površina elipse računa po formuli  $P_{elipse} = ab\pi$  a ovde se radi o  $\frac{1}{4}$  te površine.

$$I_2 = \iint_G ydzdx$$

$$4x^2 + y^2 + 4z^2 = 1 \rightarrow y^2 = 1 - 4x^2 - 4z^2 \rightarrow y = \sqrt{1 - 4x^2 - 4z^2}$$

$$I_2 = \iint_{G^+} y dz dx = \iint_D \sqrt{1-4x^2-4z^2} dz dx$$

Kako je ovde  $\frac{x^2}{\frac{1}{4}} + \frac{y^2}{1} + \frac{z^2}{\frac{1}{4}} = 1 \wedge y = 0 \rightarrow \frac{x^2}{\frac{1}{4}} + \frac{z^2}{\frac{1}{4}} = 1 \rightarrow x^2 + z^2 = \frac{1}{4}$  Odlast D će biti ( pogledajmo sliku):



$$\left. \begin{array}{l} x = r \cos \varphi \\ z = r \sin \varphi \end{array} \right\} \rightarrow |J| = r \wedge r^2 = \frac{1}{4} \rightarrow r = \frac{1}{2} \rightarrow 0 \leq r \leq \frac{1}{2} \wedge 0 \leq \varphi \leq \frac{\pi}{2}$$

$$I_2 = \iint_{G^+} y dz dx = \iint_D \sqrt{1-4x^2-4z^2} dz dx = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{1}{2}} \sqrt{1-4r^2} \cdot r dr = \dots = \frac{\pi}{24}$$

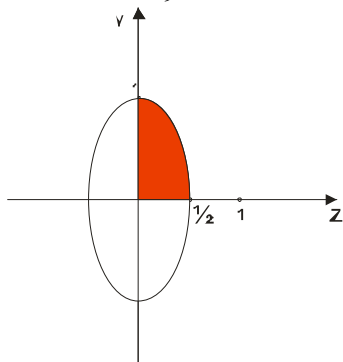
$$I_3 = \iint_G -x^2 z dy dz$$

$$4x^2 + y^2 + 4z^2 = 1 \rightarrow x^2 = \frac{1}{4}(1 - y^2 - 4z^2) \rightarrow x = \frac{1}{2}\sqrt{1 - y^2 - 4z^2}$$

$$I_3 = \iint_{G^+} -x^2 z dy dz = \iint_D -\frac{1}{4}(1 - y^2 - 4z^2) z dy dz = -\frac{1}{4} \iint_D (1 - y^2 - 4z^2) z dy dz$$

Kako je  $\frac{x^2}{\frac{1}{4}} + \frac{y^2}{1} + \frac{z^2}{\frac{1}{4}} = 1 \wedge x = 0 \rightarrow \frac{y^2}{1} + \frac{z^2}{\frac{1}{4}} = 1 \rightarrow y^2 + 4z^2 = 1$  to ćemo uzeti:

$$\left. \begin{array}{l} y = r \cos \varphi \\ z = \frac{1}{2} r \sin \varphi \end{array} \right\} \rightarrow |J| = \frac{1}{2} r \wedge r^2 = 1 \rightarrow r = 1 \rightarrow 0 \leq r \leq 1 \wedge 0 \leq \varphi \leq \frac{\pi}{2}$$





$$I_3 = -\frac{1}{4} \iint_D (1 - y^2 - 4z^2) z dy dz = -\frac{1}{4} \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 \frac{1}{2} r \sin \varphi (1 - r^2) \frac{1}{2} r dr = \dots = -\frac{1}{120}$$

Sad saberemo sva tri rešenja:

$$I = I_1 + I_2 + I_3 = \frac{\pi}{4} + \frac{\pi}{24} - \frac{1}{120} = \boxed{\frac{7\pi}{24} - \frac{1}{120}}$$

Vama za trening ostavljamo da zadatak rešite na onaj “prvi” način!

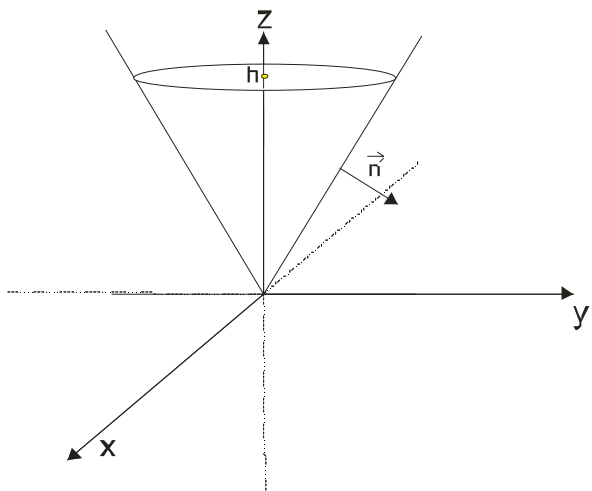
Uostalom, najbolje je da radite onako kako zahteva vaš profesor!

**4. Izračunati integral**  $I = \iint_S (y - z) dy dz + (z - x) dx dz + (x - y) dx dy$  ako je S spoljna strana površi

$$x^2 + y^2 = z^2 \wedge 0 \leq z \leq h$$

**Rešenje:**

Dakle, ovde se radi o konusu, pogledajmo sliku....



Iz  $x^2 + y^2 = z^2 \wedge 0 \leq z \leq h \rightarrow z = \sqrt{x^2 + y^2}$  pa je onda:

$$p = \frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \wedge q = \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\sqrt{1 + p^2 + q^2} = \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{1 + x^2 + y^2}}\right)^2} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{\frac{2(x^2 + y^2)}{x^2 + y^2}} = \sqrt{2}$$

Sa slike vidimo da normala na spoljnu stranu konusa gradi sa pozitivnim smerom z ose tup ugao pa je:  $\gamma > 90^\circ \rightarrow \cos \gamma < 0 \rightarrow$  uzimamo **plus** ispred korena u imeniocu  $\cos \gamma = \frac{-1}{+\sqrt{1 + p^2 + q^2}} = -\frac{1}{\sqrt{1 + p^2 + q^2}} = -\frac{1}{\sqrt{2}}$

a onda je

$$\cos \alpha = \frac{p}{+\sqrt{1+p^2+q^2}} = \frac{\frac{x}{\sqrt{x^2+y^2}}}{\sqrt{2}} =$$

$$\cos \beta = \frac{q}{+\sqrt{1+p^2+q^2}} = \frac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{2}} =$$

Sad upotrebljavamo formulu:

$$\begin{aligned} I &= \iint_S (y-z)dydz + (z-x)dx dz + (x-y)dx dy = \iint_S [(y-z)\cos \alpha + (z-x)\cos \beta + (x-y)\cos \gamma]dS = \\ &= \iint_S [(y-\sqrt{x^2+y^2})\frac{\frac{x}{\sqrt{x^2+y^2}}}{\sqrt{2}} + (\sqrt{x^2+y^2}-x)\frac{\frac{y}{\sqrt{x^2+y^2}}}{\sqrt{2}} + (x-y)(-\frac{1}{\sqrt{2}})]dS = \\ &= \iint_S [\cancel{\frac{xy}{\sqrt{x^2+y^2}} - \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} - \cancel{\frac{xy}{\sqrt{x^2+y^2}}} + (y-x)\frac{1}{\sqrt{2}}]dS = \iint_S 2(y-x)\frac{1}{\sqrt{2}}dS = \iint_S \sqrt{2}(y-x)dS = \sqrt{2}\iint_D (y-x)\sqrt{1+p^2+q^2}dxdy \end{aligned}$$

Sad spustimo problem u ravan  $z=0$  i odredimo granice:

$$D: x^2 + y^2 \leq h^2$$

$$\left. \begin{aligned} x &= r \cos \varphi \\ y &= r \sin \varphi \end{aligned} \right\} \rightarrow |J| = r \rightarrow D: \begin{cases} 0 \leq r \leq h \\ 0 \leq \varphi \leq 2\pi \end{cases}$$

$$I = \iint_S \sqrt{2}(y-x)dS = \sqrt{2} \int_0^{2\pi} d\varphi \int_0^h (r \sin \varphi - r \cos \varphi) \sqrt{2} r dr = 2 \int_0^{2\pi} (\sin \varphi - \cos \varphi) d\varphi \int_0^h r^2 dr = \dots = 0$$

[www.matematiranje.com](http://www.matematiranje.com)