

INTEGRALI ZADACI (I – DEO)

Ako je $f(x)$ neprekidna funkcija i $F'(x) = f(x)$ onda je $\int f(x)dx = F(x) + C$, gde je C proizvoljna konstanta.

Morate naučiti tablicu osnovnih integrala:

1. $\int dx = x + C$
2. $\int x dx = \frac{x^2}{2} + C$
3. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ najčešće se koristi...
4. $\int \frac{1}{x} dx = \ln|x| + C$ ili da vas ne zbuni $\int \frac{dx}{x} = \ln|x| + C$
5. $\int a^x dx = \frac{a^x}{\ln a} + C$
6. $\int e^x dx = e^x + C$
7. $\int \sin x dx = -\cos x + C$
8. $\int \cos x dx = +\sin x + C$
9. $\int \frac{1}{\sin^2 x} dx = -\operatorname{ctgx} + C$
10. $\int \frac{1}{\cos^2 x} dx = \operatorname{tgx} + C$
11. $\int \frac{1}{1+x^2} dx = \frac{\operatorname{arctgx} + C}{-\operatorname{arccotgx} + C}$ ili to jest $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C$
12. $\int \frac{1}{\sqrt{1-x^2}} dx = \frac{\arcsin x + C}{-\arccos x + C}$ ili to jest $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C$

Ovo su osnovni tablični integrali. Neki profesori dozvoljavaju da se kao tablični koriste i :

13. $\int \frac{dx}{1-x^2} = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C$ odnosno $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C$ to jest $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$
14. $\int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln \left| x + \sqrt{x^2 \pm 1} \right| + C$ odnosno $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C$

Primeri:

1. $\int x^5 dx = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C$ kao 3. tablični $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

2. $\int 7^x dx = \frac{7^x}{\ln 7} + C$ kao 5. tablični $\int a^x dx = \frac{a^x}{\ln a} + C$

3. $\int \sqrt{x} dx$ = pogledamo i vidimo da ga ovaj integral nema u tablici osnovnih integrala... Ideja je da se kod ovakvih integrala iskoristi pravilo za stepenovanje $\boxed{\sqrt[m]{x^n} = x^{\frac{n}{m}}}$, odnosno $\boxed{\sqrt{x} = x^{\frac{1}{2}}}$. Na ovaj način se integral svede na najčešće upotrebljavani tablični $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.

Dakle:

$$\int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2x^{\frac{3}{2}}}{3} + C$$

4. $\int \frac{1}{x^{12}} dx$ = I ovaj ga nema u tablici... Za njega ćemo upotrebiti pravilo za stepenovanje, da je $\frac{1}{x^n} = x^{-n}$...

$$\int \frac{1}{x^{12}} dx = \int x^{-12} dx = \frac{x^{-12+1}}{-12+1} + C = \frac{x^{-11}}{-11} + C = -\frac{1}{11x^{11}} + C$$

Najbolje je da se mi podsetimo svih pravila za stepenovanje i korenovanje:

1) $a^0 = 1$

2) $a^{-n} = \frac{1}{a^n}$

3) $a^m \cdot a^n = a^{m+n}$

4) $a^m : a^n = a^{m-n}$

5) $(a^m)^n = a^{m \cdot n}$

6) $(a \cdot b)^n = a^n \cdot b^n$

7) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

8) $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

- 1) $\sqrt[n]{a^m} = a^{\frac{m}{n}}$
- 2) $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
- 3) $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{a : b}$
- 4) $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$
- 5) $\sqrt[n]{\sqrt[m]{a}} = \sqrt[n \cdot m]{a}$
- 6) $\sqrt[n]{\sqrt[m]{a^m}} = \sqrt[n]{a}$

Nastavimo sa primerima...

5. $\int x \cdot \sqrt[3]{x} dx =$ Upotrebimo pravila za stepen i koren da "pripremimo" podintegralnu funkciju :

$$x \cdot \sqrt[3]{x} = x^1 \cdot x^{\frac{1}{3}} = x^{1+\frac{1}{3}} = x^{\frac{4}{3}}$$

$$\int x \cdot \sqrt[3]{x} dx = \int x^{\frac{4}{3}} dx = \frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1} + C = \frac{x^{\frac{7}{3}}}{\frac{7}{3}} + C = \frac{3x^{\frac{7}{3}}}{7} + C$$

$$6. \quad \int 5^x \cdot 3^{-x} dx = \int 5^x \cdot \frac{1}{3^x} dx = \int \left(\frac{5}{3}\right)^x dx = \frac{\left(\frac{5}{3}\right)^x}{\ln\left(\frac{5}{3}\right)} + C$$

7.

$$\int \sqrt{x\sqrt{x\sqrt{x}}} dx = ?$$

"Spakujemo" podintegralnu funkciju $\sqrt{x\sqrt{x\sqrt{x}}} = \sqrt{x\sqrt{\sqrt{x^2} \cdot x}} = \sqrt{x\sqrt[4]{x^3}} = \sqrt{\sqrt[4]{x^4 \cdot x^3}} = \sqrt[8]{x^7} = x^{\frac{7}{8}}$

$$\int x^{\frac{7}{8}} dx = \frac{x^{\frac{7}{8}+1}}{\frac{7}{8}+1} + C = \frac{x^{\frac{15}{8}}}{\frac{15}{8}} + C = \frac{8x^{\frac{15}{8}}}{15} + C$$

Da se upoznamo i sa osnovnim svojstvima neodređenog integrala:

$$1) \quad \boxed{\int A \cdot f(x) dx = A \cdot \int f(x) dx} \quad \text{gde je A konstanta (broj)}$$

Dakle, slično kao i kod izvoda, konstanta (broj) izlazi ispred integrala...

Primeri:

8.

$$\int 4x^3 dx = ?$$

$$\int 4x^3 dx = 4 \int x^3 dx = 4 \cdot \frac{x^4}{4} + C = x^4 + C$$

9.

$$\int \frac{1}{4x} dx = ?$$

$$\int \frac{1}{4x} dx = \frac{1}{4} \cdot \int \frac{1}{x} dx = \frac{1}{4} \cdot \ln|x| + C$$

10.

$$\int 2\pi \sin x dx = ?$$

$$\int 2\pi \sin x dx = 2\pi \cdot \int \sin x dx = 2\pi \cdot (-\cos x) + C = -2\pi \cos x + C$$

$$2) \quad \boxed{\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx}$$

Opet slično kao kod izvoda: Ako imamo zbir ili razliku više funkcija od svake tražimo posebno integral...

11.

$$\int (4x^2 + 2x - 3) dx = ?$$

$$\int (4x^2 + 2x - 3) dx = \int 4x^2 dx + \int 2x dx - \int 3 dx = \text{konstante izbacimo ispred integrala...}$$

$$= 4 \int x^2 dx + 2 \int x dx - 3 \int dx$$

$$= 4 \frac{x^3}{3} + 2 \frac{x^2}{2} - 3x + C = \boxed{4 \frac{x^3}{3} + x^2 - 3x + C}$$

12.

$$\int (5 \cos x + \frac{1}{3} e^x - 2x^3 + \frac{4}{x} - \frac{23}{\sin^2 x} + 2 \cdot 5^x) dx = ?$$

$$\begin{aligned} \int (5 \cos x + \frac{1}{3} e^x - 2x^3 + \frac{4}{x} - \frac{23}{\sin^2 x} + 2 \cdot 5^x) dx &= 5 \int \cos x dx + \frac{1}{3} \int e^x dx - 2 \int x^3 dx + 4 \int \frac{1}{x} dx - 23 \int \frac{dx}{\sin^2 x} + 2 \cdot \int 5^x dx = \\ &= 5 \sin x + \frac{1}{3} e^x - 2 \frac{x^4}{4} + 4 \ln|x| - 23(-\operatorname{ctgx}) + 2 \frac{5^x}{\ln 5} + C \end{aligned}$$

13.

$$\int \frac{x-2}{x^3} dx = ?$$

Kod ovog i sličnih integrala ćemo upotrebiti $\boxed{\frac{A \pm B}{C} = \frac{A}{C} \pm \frac{B}{C}}$

$$\begin{aligned} \int \frac{x-2}{x^3} dx &= \int \left(\frac{x}{x^3} - \frac{2}{x^3} \right) dx = \int (x^{-2} - 2x^{-3}) dx = \int x^{-2} dx - 2 \int x^{-3} dx \\ &= \frac{x^{-2+1}}{-2+1} - 2 \frac{x^{-3+1}}{-3+1} + C \\ &= \boxed{-\frac{1}{x} + \frac{1}{x^2} + C} \end{aligned}$$

14.

$$\int \frac{2^{x+1} - 5^{x-1}}{10^x} dx = ?$$

Da prisredimo najpre malo podintegralnu funkciju...

$$\frac{2^{x+1} - 5^{x-1}}{10^x} = \frac{2^x \cdot 2^1 - \frac{5^x}{5^1}}{10^x} = \frac{2^x \cdot 2^1}{10^x} - \frac{5^x}{10^x} = 2 \cdot \left(\frac{2}{10} \right)^x - \frac{1}{5} \cdot \left(\frac{5}{10} \right)^x = 2 \cdot \left(\frac{1}{5} \right)^x - \frac{1}{5} \cdot \left(\frac{1}{2} \right)^x$$

$$\begin{aligned} \int \frac{2^{x+1} - 5^{x-1}}{10^x} dx &= \int \left[2 \cdot \left(\frac{1}{5} \right)^x - \frac{1}{5} \cdot \left(\frac{1}{2} \right)^x \right] dx = \int 2 \cdot \left(\frac{1}{5} \right)^x dx - \int \frac{1}{5} \cdot \left(\frac{1}{2} \right)^x dx = \\ &= 2 \cdot \int \left(\frac{1}{5} \right)^x dx - \frac{1}{5} \cdot \int \left(\frac{1}{2} \right)^x dx = 2 \cdot \frac{\left(\frac{1}{5} \right)^x}{\ln \frac{1}{5}} - \frac{1}{5} \frac{\left(\frac{1}{2} \right)^x}{\ln \frac{1}{2}} + C \end{aligned}$$

15.

$\int \frac{x^2}{x^2+1} dx = ?$ Ovo je tip integrala koji najlakše rešavamo malim "trikom" (dodamo 1 i oduzmemo 1)

$$\begin{aligned} \int \frac{x^2}{x^2+1} dx &= \int \frac{x^2+1-1}{x^2+1} dx = \int \frac{x^2+1}{x^2+1} dx - \int \frac{1}{x^2+1} dx = \int \frac{\cancel{x^2+1}}{\cancel{x^2+1}} dx - \int \frac{1}{x^2+1} dx \\ &= \int dx - \int \frac{1}{x^2+1} dx = \boxed{x - \arctg x + C} \end{aligned}$$

I ovde će nam trebati znanje iz trigonometrije. Da se podsetimo nekih najvažnijih formula:

Osnovni trigonometrijski identiteti

$$1) \sin^2 \alpha + \cos^2 \alpha = 1$$

$$2) \operatorname{tg} \alpha = \frac{\sin \alpha}{\cos \alpha}$$

$$3) \operatorname{ctg} \alpha = \frac{\cos \alpha}{\sin \alpha}$$

$$4) \operatorname{tg} \alpha \cdot \operatorname{ctg} \alpha = 1$$

Adicione formule

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\operatorname{tg}(\alpha + \beta) = \frac{\operatorname{tg} \alpha + \operatorname{tg} \beta}{1 - \operatorname{tg} \alpha \cdot \operatorname{tg} \beta}$$

$$\operatorname{ctg}(\alpha + \beta) = \frac{\operatorname{ctg} \alpha \cdot \operatorname{ctg} \beta - 1}{\operatorname{ctg} \beta + \operatorname{ctg} \alpha}$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\operatorname{tg}(\alpha - \beta) = \frac{\operatorname{tg} \alpha - \operatorname{tg} \beta}{1 + \operatorname{tg} \alpha \cdot \operatorname{tg} \beta}$$

$$\operatorname{ctg}(\alpha - \beta) = \frac{\operatorname{ctg} \alpha \cdot \operatorname{ctg} \beta + 1}{\operatorname{ctg} \beta - \operatorname{ctg} \alpha}$$

Polovina ugla

$$1. \sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}} \quad \text{ili} \quad 2 \sin^2 \frac{\alpha}{2} = 1 - \cos \alpha \quad \text{odnosno} \quad \boxed{\sin^2 x = \frac{1 - \cos 2x}{2}}$$

$$2. \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \quad \text{ili} \quad 2 \cos^2 \frac{\alpha}{2} = 1 + \cos \alpha \quad \text{odnosno} \quad \boxed{\cos^2 x = \frac{1 + \cos 2x}{2}}$$

$$3. \operatorname{tg} \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

$$4. \operatorname{ctg} \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}}$$

Transformacije zbira i razlike u proizvod

$$1. \sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$2. \sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$3. \cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$4. \cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$5. \operatorname{tg} \alpha \pm \operatorname{tg} \beta = \frac{\sin(\alpha \pm \beta)}{\cos \alpha \cos \beta}$$

$$6. \operatorname{ctg} \alpha \pm \operatorname{ctg} \beta = \frac{\sin(\alpha \pm \beta)}{\sin \alpha \sin \beta}$$

Dvostruki ugao

$$1. \sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$2. \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$3. \operatorname{tg} 2\alpha = \frac{2 \operatorname{tg} \alpha}{1 - \operatorname{tg}^2 \alpha}$$

$$4. \operatorname{ctg} 2\alpha = \frac{\operatorname{ctg}^2 \alpha - 1}{2 \operatorname{ctg} \alpha}$$

16.

$$\int \frac{\cos 2x}{\sin^2 x \cdot \cos^2 x} dx = ? \quad \text{treba nam formula } \boxed{\cos 2x = \cos^2 x - \sin^2 x}$$

$$\int \frac{\cos 2x}{\sin^2 x \cdot \cos^2 x} dx = \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cdot \cos^2 x} dx = \text{rastavimo na dva integrala...}$$

$$\int \frac{\cos^2 x}{\sin^2 x \cdot \cos^2 x} dx - \int \frac{\sin^2 x}{\sin^2 x \cdot \cos^2 x} dx = \text{skratimo...}$$

$$\int \frac{\cancel{\cos^2 x}}{\sin^2 x \cdot \cancel{\cos^2 x}} dx - \int \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x} \cdot \cos^2 x} dx = \text{i dobijamo dva tablična integrala...}$$

$$\int \frac{1}{\sin^2 x} dx - \int \frac{1}{\cos^2 x} dx = \boxed{-\operatorname{ctgx} - \operatorname{tgx} + C}$$

17.

$$\int \frac{dx}{\sin^2 x \cdot \cos^2 x} = ? \quad \text{treba nam formula } \boxed{\sin^2 x + \cos^2 x = 1}$$

$$\int \frac{1 \cdot dx}{\sin^2 x \cdot \cos^2 x} = \text{uz } dx \text{ je } 1, \text{ zar ne?} = \int \frac{1}{\sin^2 x \cdot \cos^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos^2 x} dx = \text{rastavimo na dva integrala...}$$

$$\int \frac{\sin^2 x}{\sin^2 x \cdot \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cdot \cos^2 x} dx = \text{skratimo...}$$

$$\int \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x} \cdot \cos^2 x} dx + \int \frac{\cancel{\cos^2 x}}{\sin^2 x \cdot \cancel{\cos^2 x}} dx = \text{i dobijamo dva tablična integrala...}$$

$$\int \frac{1}{\cos^2 x} dx - \int \frac{1}{\sin^2 x} dx = \boxed{\operatorname{tgx} + \operatorname{ctgx} + C}$$

18.

$$\int \operatorname{tg}^2 x dx = ?$$

$$\text{Ovde koristimo } \operatorname{tgx} = \frac{\sin x}{\cos x}$$

$$\int \operatorname{tg}^2 x dx = \int \frac{\sin^2 x}{\cos^2 x} dx = \text{kako je } \sin^2 \alpha + \cos^2 \alpha = 1 \rightarrow \boxed{\sin^2 \alpha = 1 - \cos^2 \alpha}, \text{ pa je}$$

$$= \int \frac{1 - \cos^2 \alpha}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} dx - \int \frac{\cancel{\cos^2 \alpha}}{\cancel{\cos^2 x}} dx = \int \frac{1}{\cos^2 x} dx - \int dx = \boxed{\operatorname{tgx} - x + C}$$

19.

$$\int \frac{18x^2 - 2}{3x - 1} dx = ?$$

Probamo da sredimo podintegralnu funkciju, ako neće, mora se koristiti neki drugi trik(druga metoda)...

$$\frac{18x^2 - 2}{3x - 1} = \frac{2(9x^2 - 1)}{3x - 1} = \frac{2\cancel{(3x - 1)}(3x + 1)}{\cancel{3x - 1}} = 2(3x + 1) = 6x + 2 \quad \text{Sad je već lakše...}$$

$$\int \frac{18x^2 - 2}{3x - 1} dx = \int (6x + 2) dx = 6 \int x dx + 2 \int dx = 6 \frac{x^2}{2} + 2x + C = \boxed{3x^2 + 2x + C}$$

20.

$$\int \frac{4 - x}{2 + \sqrt{x}} dx = ?$$

$$\frac{4 - x}{2 + \sqrt{x}} = \frac{2^2 - (\sqrt{x})^2}{2 + \sqrt{x}} = \frac{(2 - \sqrt{x})(2 + \sqrt{x})}{2 + \sqrt{x}} = \frac{(2 - \sqrt{x})\cancel{(2 + \sqrt{x})}}{\cancel{2 + \sqrt{x}}} = 2 - \sqrt{x} = 2 - x^{\frac{1}{2}}$$

$$\int \frac{4 - x}{2 + \sqrt{x}} dx = \int (2 - x^{\frac{1}{2}}) dx = \int 2 dx - \int x^{\frac{1}{2}} dx = 2x - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \boxed{2x - \frac{2x^{\frac{3}{2}}}{3} + C}$$

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