

Dvostruki integrali-rešavanje

Primer 1.

Izračunati integral $\iint_D \frac{x^2}{1+y^2} dx dy$ ako je $0 \leq x \leq 1$ i $0 \leq y \leq 1$

Rešenje:

Ovde su nam odmah date granice integrala pa ne moramo crtati sliku i odredjivati ih.

Uvek je pitanje da li je lakše raditi prvo po x pa po y ili obrnuto...

Pogledajte najpre dati dvojni integral, razmislite malo pa tek onda krenite sa radom...

Mi ćemo prvo rešavati ovaj integral po x pa onda po y...

$$\iint_D \frac{x^2}{1+y^2} dx dy = \int_0^1 dy \int_0^1 \frac{x^2}{1+y^2} dx \quad \text{neki profesori vole da zapišu i ovako:}$$

$$\iint_D \frac{x^2}{1+y^2} dx dy = \int_0^1 \left(\int_0^1 \frac{x^2}{1+y^2} dx \right) dy \quad \text{Vi naravno radite kao što zahteva vaš profesor...}$$

Dakle, najpre rešavamo integral u zagradi. On je “po x” pa ovde **y tretiramo kao konstantu!**

$$\iint_D \frac{x^2}{1+y^2} dx dy = \int_0^1 \left(\int_0^1 \frac{x^2}{1+y^2} dx \right) dy = \int_0^1 \frac{1}{1+y^2} \left(\int_0^1 x^2 dx \right) dy =$$

Rešićemo ga “na stranu” ...

$$\int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}$$

Sad ovo rešenje ubacimo u dvojni integral:

$$\begin{aligned} \iint_D \frac{x^2}{1+y^2} dx dy &= \int_0^1 \left(\int_0^1 \frac{x^2}{1+y^2} dx \right) dy = \int_0^1 \frac{1}{1+y^2} \left(\int_0^1 x^2 dx \right) dy = \int_0^1 \frac{1}{1+y^2} dy = \\ &= \frac{1}{3} \int_0^1 \frac{1}{1+y^2} dy = \frac{1}{3} \arctg y \Big|_0^1 = \frac{1}{3} (\arctg 1 - \arctg 0) = \frac{1}{3} \left(\frac{\pi}{4} - 0 \right) = \boxed{\frac{\pi}{12}} \end{aligned}$$

I evo rešenja.

Primer 2.

Izračunati integral $\iint_D x \sin(x+y) dx dy$ ako je $0 \leq x \leq \pi$ i $0 \leq y \leq \frac{\pi}{2}$

Rešenje:

I ovde odmah imamo granice, pa slika ne treba...

$$\iint_D x \sin(x+y) dx dy = \int_0^\pi x dx \int_0^{\frac{\pi}{2}} \sin(x+y) dy = \int_0^\pi x \left(\int_0^{\frac{\pi}{2}} \sin(x+y) dy \right) dx$$

Integral u zagradi rešimo na stranu:

$$\int_0^{\frac{\pi}{2}} \sin(x+y) dy = -\cos(x+y) \Big|_0^{\frac{\pi}{2}} = -[\cos(x+\frac{\pi}{2}) - \cos(x+0)] = -[\cos(x+\frac{\pi}{2}) - \cos x]$$

Iz trigonometrije znamo da je $\cos(x+\frac{\pi}{2}) = -\sin x$ pa je

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin(x+y) dy &= -\cos(x+y) \Big|_0^{\frac{\pi}{2}} = -[\cos(x+\frac{\pi}{2}) - \cos(x+0)] = -[\cos(x+\frac{\pi}{2}) - \cos x] = -[-\sin x - \cos x] \\ &= \sin x + \cos x \end{aligned}$$

Vratimo se u dvojni integral:

$$\iint_D x \sin(x+y) dx dy = \int_0^\pi x dx \int_0^{\frac{\pi}{2}} \sin(x+y) dy = \int_0^\pi x \left(\int_0^{\frac{\pi}{2}} \sin(x+y) dy \right) dx = \int_0^\pi x(\sin x + \cos x) dx$$

Ovde imamo parcijalnu integraciju, za oba integrala...

$$\begin{aligned} \int_0^\pi x \sin x dx &= \left| \begin{array}{ll} x = u & \sin x dx = dv \\ dx = du & -\cos x = v \end{array} \right| = -x \cos x - \int (-\cos x) dx = \\ &= (-x \cos x + \sin x) \Big|_0^\pi = (-\pi \cos \pi + \sin \pi) - (-0 \cos 0 + \sin 0) = -\pi(-1) = \pi \end{aligned}$$

$$\int_0^{\pi} x \cos x dx = \left| \begin{array}{ll} x = u & \cos x dx = dv \\ dx = du & \sin x = v \end{array} \right| = x \sin x - \int \sin x dx =$$

$$= (x \sin x + \cos x) \Big|_0^{\pi} = (\pi \sin \pi + \cos \pi) - (0 \sin 0 + \cos 0) = -1 - 1 = -2$$

Rešenje će biti:

$$\iint_D x \sin(x+y) dx dy = \int_0^{\pi} x dx \int_0^{\frac{\pi}{2}} \sin(x+y) dy = \int_0^{\pi} x \left(\int_0^{\frac{\pi}{2}} \sin(x+y) dy \right) dx = \int_0^{\pi} x (\sin x + \cos x) dx = \boxed{\pi - 2}$$

Kao što ste videli ovaj dvojni integral smo prvo rešavali po y pa onda po x.

Da li bi bilo lakše da smo išli obrnuto?

Da vidimo:

II način

$$\iint_D x \sin(x+y) dx dy = \int_0^{\frac{\pi}{2}} \left(\int_0^{\pi} x \sin(x+y) dx \right) dy$$

Integral u zagradi rešićemo na stranu, kao neodređeni, pa ćemo mu dodati granice:

$$\int_0^{\pi} x \sin(x+y) dx = ?$$

$$\int x \sin(x+y) dx = \left| \begin{array}{ll} x = u & \sin(x+y) dx = dv \\ dx = du & -\cos(x+y) = v \end{array} \right| = -x \cos(x+y) - \int [-\cos(x+y)] dx =$$

$$= -x \cos(x+y) + \sin(x+y)$$

$$\int_0^{\pi} x \sin(x+y) dx = -x \cos(x+y) + \sin(x+y) \Big|_0^{\pi} = [-\pi \cos(\pi+y) + \sin(\pi+y)] - [-0 \cos(0+y) + \sin(0+y)] =$$

$$= -\pi \cos(\pi+y) + \sin(\pi+y) - \sin y$$

$$\iint_D x \sin(x+y) dx dy = \int_0^{\frac{\pi}{2}} \left(\int_0^{\pi} x \sin(x+y) dx \right) dy = \int_0^{\frac{\pi}{2}} (-\pi \cos(\pi+y) + \sin(\pi+y) - \sin y) dy =$$

$$= \left(-\pi \sin(\pi+y) - \cos(\pi+y) + \cos y \right) \Big|_0^{\frac{\pi}{2}} =$$

$$= \left(-\pi \sin\left(\pi + \frac{\pi}{2}\right) - \cos\left(\pi + \frac{\pi}{2}\right) + \cos \frac{\pi}{2} \right) - \left(-\pi \sin(\pi+0) - \cos(\pi+0) + \cos 0 \right) =$$

$$= \pi - 0 + 0 - (0 + 1 + 1) = \boxed{\pi - 2}$$

Možda malo brže... Bitno je da je rešenje dobro!

Primer 3.

Izračunati integral $\iint_D xy^2 dx dy$ ako je oblast integracije ograničena parabolom $y^2 = 2x$ i pravom $x = \frac{1}{2}$

Rešenje:

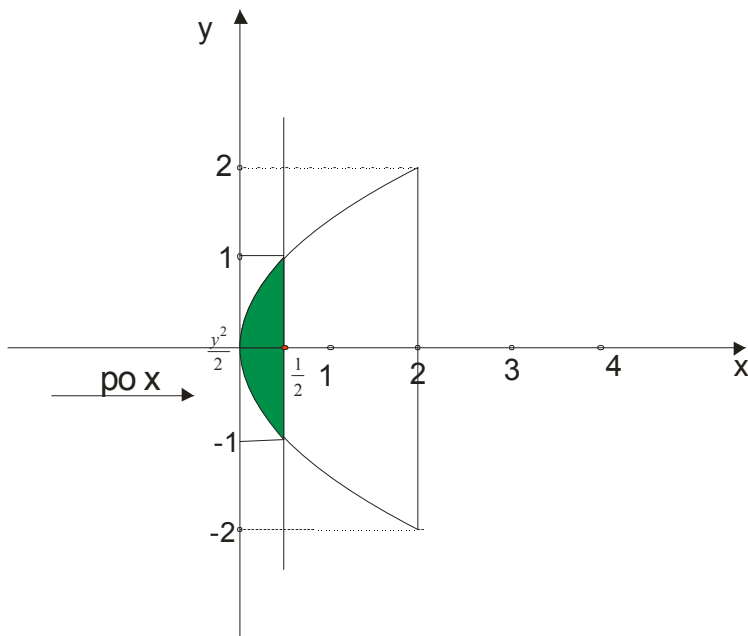
Najpre ćemo odrediti preseke i nacrtati sliku:

Presek odredjujemo rešavajući sistem jednačina.

$$y^2 = 2x$$

$$x = \frac{1}{2}$$

$$y^2 = 2 \cdot \frac{1}{2} \rightarrow y^2 = 1 \rightarrow y = \pm 1 \rightarrow M\left(\frac{1}{2}, -1\right) \wedge N\left(\frac{1}{2}, 1\right)$$



Slika nam pomaže da odredimo odlast $D: \begin{cases} -1 \leq y \leq 1 \\ \frac{y^2}{2} \leq x \leq \frac{1}{2} \end{cases}$ (pogledajte prethodni fajl)

Sad imamo:

$$\iint_D xy^2 dx dy = \int_{-1}^1 dy \int_{\frac{y^2}{2}}^{\frac{1}{2}} xy^2 dx = \int_{-1}^1 \left(\int_{\frac{y^2}{2}}^{\frac{1}{2}} xy^2 dx \right) dy \quad \text{Prvo rešavamo integral u zagradi:}$$

$$\int_{\frac{y^2}{2}}^{\frac{1}{2}} xy^2 dx = y^2 \cdot \int_{\frac{y^2}{2}}^{\frac{1}{2}} x dx = y^2 \cdot \left[\frac{x^2}{2} \right]_{\frac{y^2}{2}}^{\frac{1}{2}} = y^2 \cdot \left[\frac{\left(\frac{1}{2}\right)^2}{2} - \frac{\left(\frac{y^2}{2}\right)^2}{2} \right] = \frac{y^2}{2} \left[\frac{1}{4} - \frac{y^4}{4} \right] = \frac{1}{8} [y^2 - y^6]$$

Vračamo se u dvojni integral:

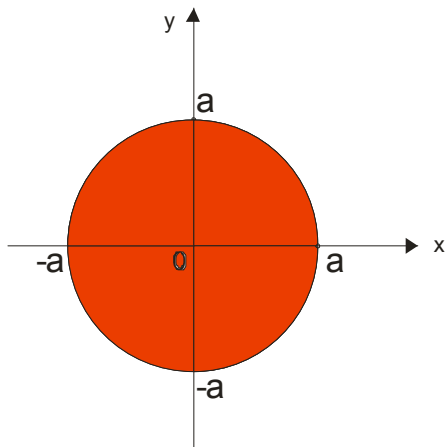
$$\begin{aligned} \iint_D xy^2 dx dy &= \int_{-1}^1 dy \int_{\frac{y^2}{2}}^{\frac{1}{2}} xy^2 dx = \int_{-1}^1 \left(\frac{1}{8} [y^2 - y^6] \right) dy = \\ &= \frac{1}{8} \left[\frac{y^3}{3} - \frac{y^7}{7} \right] \Big|_{-1}^1 = \frac{1}{8} \left\{ \left[\frac{1^3}{3} - \frac{1^7}{7} \right] - \left[\frac{(-1)^3}{3} - \frac{(-1)^7}{7} \right] \right\} = \frac{1}{8} \cdot \frac{8}{21} = \boxed{\frac{1}{21}} \end{aligned}$$

Primer 4.

Izračunati $\iint_D \sqrt{x^2 + y^2} dx dy$ ako je oblast D zadata sa $x^2 + y^2 \leq a^2$

Rešenje:

Pogledajmo sliku:



U ovakvim slučajevima, kad je zadata kružnica, zgodno je preći na polarne koordinate:

$$x = r \cos \varphi$$

$$y = r \sin \varphi$$

$$|J| = r$$

I onda se integral rešava:

$$\iint_D z(x, y) dx dy = \iint_{D'} z(r \cos \varphi, r \sin \varphi) |J| dr d\varphi = \int_{\varphi_1}^{\varphi_2} d\varphi \int_0^r z(r \cos \varphi, r \sin \varphi) r dr$$

Najpre da odredimo granice:

$$x^2 + y^2 = a^2$$

$$(r \cos \varphi)^2 + (r \sin \varphi)^2 = a^2$$

$$r^2 (\cos^2 \varphi + \sin^2 \varphi) = a^2 \quad \text{znamo da je } \cos^2 \varphi + \sin^2 \varphi = 1$$

$$r^2 = a^2 \rightarrow r = a$$

Dakle r ide od 0 da a .

Pošto nam ovde treba ceo krug, jasno je da $0 \leq \varphi \leq 2\pi$

$$\text{Imamo dakle da je } D = \begin{cases} 0 \leq r \leq a \\ 0 \leq \varphi \leq 2\pi \end{cases} \quad \text{pa je :}$$

$$\iint_D \sqrt{x^2 + y^2} dx dy = \int_0^{2\pi} d\varphi \int_0^a \sqrt{r^2} r dr = \int_0^{2\pi} \left(\int_0^a \sqrt{r^2} r dr \right) d\varphi = \int_0^{2\pi} \left(\int_0^a r^2 dr \right) d\varphi =$$

Kao i uvek, integral u zagradi rešimo posebno...

$$\int_0^a r^2 dr = \frac{r^3}{3} \Big|_0^a = \frac{a^3}{3}$$

Sad imamo:

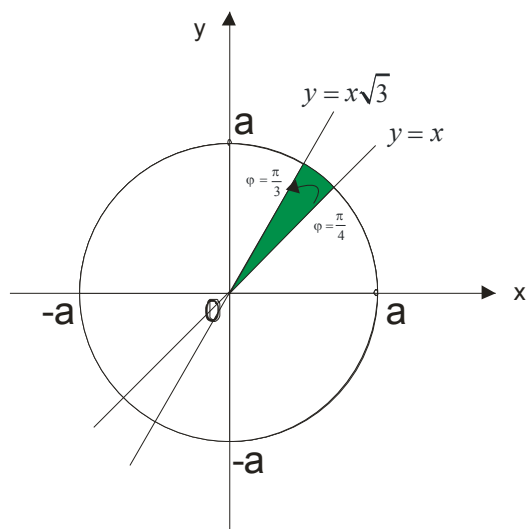
$$\begin{aligned} \iint_D \sqrt{x^2 + y^2} dx dy &= \int_0^{2\pi} d\varphi \int_0^a \sqrt{r^2} r dr = \int_0^{2\pi} \left(\int_0^a \sqrt{r^2} r dr \right) d\varphi = \int_0^{2\pi} \left(\int_0^a r^2 dr \right) d\varphi = \\ &= \int_0^{2\pi} \frac{a^3}{3} d\varphi = \frac{a^3}{3} \int_0^{2\pi} d\varphi = \frac{a^3}{3} \cdot \varphi \Big|_0^{2\pi} = \frac{a^3}{3} (2\pi - 0) = \boxed{\frac{2\pi a^3}{3}} \end{aligned}$$

Primer 5.

Izračunati $\iint_D \sqrt{a^2 - x^2 - y^2} dx dy$ ako je oblast D ograničena sa $x^2 + y^2 = a^2$, $y = x$, $y = x\sqrt{3}$ u prvom kvadrantu.

Rešenje:

Nacrtajmo sliku:



Koristićemo polarne koordinate:

$$x = r \cos \varphi$$

$$y = r \sin \varphi$$

$$|J| = r$$

Odredimo granice za r i φ :

$$x^2 + y^2 = a^2$$

$$(r \cos \varphi)^2 + (r \sin \varphi)^2 = a^2$$

$$r^2 (\cos^2 \varphi + \sin^2 \varphi) = a^2$$

$$r^2 = a^2 \rightarrow r = a$$

Dakle: r ide od 0 do a .

Iz pravih $y = x$, $y = x\sqrt{3}$ ćemo odrediti odakle dokle ide ugao φ

Da je podsetimo :

Prava $y = kx + n$ ima koeficijent pravca $k = \tan \varphi$.

Iz prave $y = x$ je $k=1$ pa je $\tan \varphi = 1 \rightarrow \varphi = \frac{\pi}{4}$

Iz prave $y = x\sqrt{3}$ je $k = \sqrt{3}$ pa je $\tan \varphi = \sqrt{3} \rightarrow \varphi = \frac{\pi}{3}$

Dobili smo dakle da je :

$$D' = \begin{cases} 0 \leq r \leq a \\ \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{3} \end{cases}$$

Da rešimo sada integral:

$$\iint_D \sqrt{a^2 - x^2 - y^2} dx dy = \iint_D \sqrt{a^2 - (x^2 + y^2)} dx dy = \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} d\varphi \int_0^a \sqrt{a^2 - r^2} \cdot r dr$$

$$\int_0^a \sqrt{a^2 - r^2} \cdot r dr \quad \text{ćemo rešiti na stranu i najpre bez granica...}$$

$$\int \sqrt{a^2 - r^2} \cdot r dr = \left| \begin{array}{l} a^2 - r^2 = t^2 \\ -2r dr = 2t dt \\ r dr = -t dt \end{array} \right| = \int \sqrt{t^2} (-t) dt = -\int t^2 dt = -\frac{t^3}{3} = -\frac{(\sqrt{a^2 - r^2})^3}{3}$$

$$\int_0^a \sqrt{a^2 - r^2} \cdot r dr = -\frac{(\sqrt{a^2 - r^2})^3}{3} \Big|_0^a = -\left[\frac{(\sqrt{a^2 - a^2})^3}{3} - \frac{(\sqrt{a^2 - 0^2})^3}{3} \right] = -\left[-\frac{a^3}{3} \right] = \frac{a^3}{3}$$

$$\iint_D \sqrt{a^2 - x^2 - y^2} dx dy = \iint_D \sqrt{a^2 - (x^2 + y^2)} dx dy = \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} d\varphi \int_0^a \sqrt{a^2 - r^2} \cdot r dr =$$

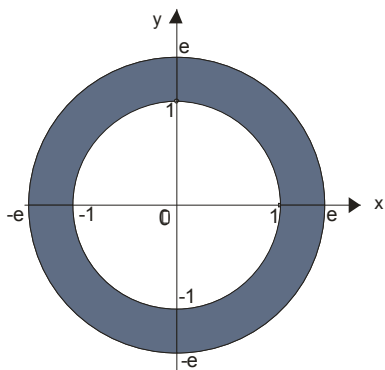
$$= \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \frac{a^3}{3} d\varphi = \frac{a^3}{3} \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} d\varphi = \frac{a^3}{3} \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \frac{a^3}{3} \frac{\pi}{12} = \boxed{\frac{a^3 \pi}{36}}$$

Primer 6.

Izračunati $\iint_D \frac{\ln(x^2 + y^2)}{x^2 + y^2} dx dy$ ako je oblast D između krugova $x^2 + y^2 = 1$ i $x^2 + y^2 = e^2$

Rešenje:

Slika:



Uzimamo polarne koordinate:

$x = r \cos \varphi$	$x^2 + y^2 = 1$	$x^2 + y^2 = e^2$
$y = r \sin \varphi$	$(r \cos \varphi)^2 + (r \sin \varphi)^2 = 1$	$(r \cos \varphi)^2 + (r \sin \varphi)^2 = e^2$
$ J = r$	$r^2 (\cos^2 \varphi + \sin^2 \varphi) = 1$	$r^2 (\cos^2 \varphi + \sin^2 \varphi) = e^2$
	$r^2 = 1 \rightarrow r = 1$	$r^2 = e^2 \rightarrow r = e$

Dakle, imamo da $1 \leq r \leq e$

Sa slike vidimo da ugao uzima pun krug $0 \leq \varphi \leq 2\pi$.

$$\iint_D \frac{\ln(x^2 + y^2)}{x^2 + y^2} dx dy = \int_0^{2\pi} d\varphi \int_1^e \frac{\ln r^2}{r^2} \cdot \cancel{r} dr = \int_0^{2\pi} d\varphi \int_1^e \frac{2 \ln r}{r} dr$$

$$\int_1^e \frac{2 \ln r}{r} dr = 2 \int_1^e \frac{\ln r}{r} dr$$

$$\int \frac{\ln r}{r} dr = \left| \begin{array}{l} \ln r = t \\ \frac{1}{r} dr = dt \end{array} \right| = \int t dt = \frac{t^2}{2} = \frac{\ln^2 r}{2}$$

$$\int_1^e \frac{2 \ln r}{r} dr = 2 \int_1^e \frac{\ln r}{r} dr = 2 \left| \frac{\ln^2 r}{2} \right|_1^e = \ln^2 e - \ln^2 1 = 1 - 0 = 1$$

$$\iint_D \frac{\ln(x^2 + y^2)}{x^2 + y^2} dx dy = \int_0^{2\pi} d\varphi \int_1^e \frac{\ln r^2}{r^2} \cdot \cancel{r} dr = \int_0^{2\pi} d\varphi \int_1^e \frac{2 \ln r}{r} dr = \int_0^{2\pi} d\varphi = 2\pi - 0 = \boxed{2\pi}$$

Primer 7.

Izračunati integral $\iint_D xy dx dy$, gde je oblast D ograničena Ox osom i lukovima krugova:

$$x^2 + y^2 = 1 \quad \text{i} \quad x^2 + y^2 - 2x = 0 \quad \text{u prvom kvadrantu.}$$

Rešenje:

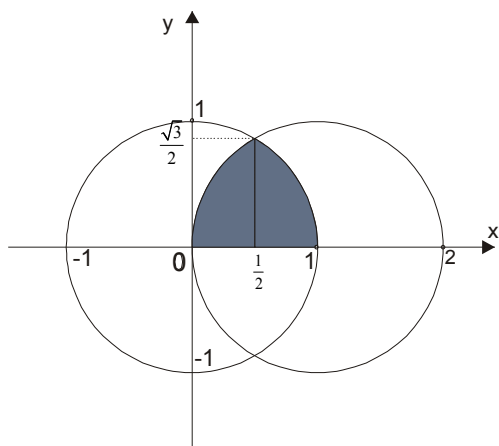
Da spakujemo drugu kružnicu, nadjemo preseke i nacrtamo sliku:

$$x^2 + y^2 - 2x = 0$$

$$x^2 - 2x + 1 - 1 + y^2 = 0$$

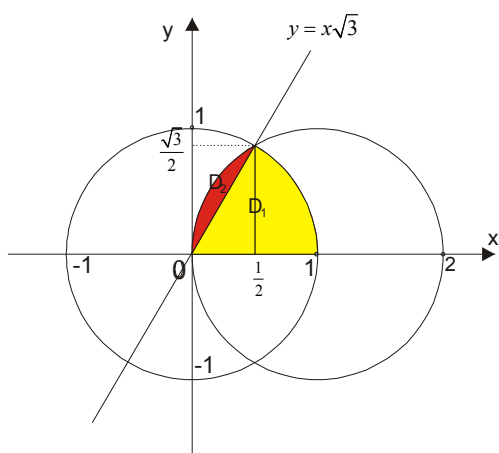
$$(x-1)^2 + y^2 = 1$$

$$\text{Presek je: } x^2 + y^2 - 2x = 0 \wedge x^2 + y^2 = 1 \rightarrow 1 - 2x = 0 \rightarrow x = \frac{1}{2} \rightarrow y = \frac{\sqrt{3}}{2}$$



Ovu oblast moramo podeliti na dva dela:

Prava koja prolazi kroz tačku preseka krugova i koordinatni početak je $y = \sqrt{3}x$ (kao jednačina prave kroz dve tačke, pogledajte prethodni fajl)



Opet ćemo preći na polarne koordinate:

$$x = r \cos \varphi$$

$$y = r \sin \varphi$$

$$|J| = r$$

Za oblast D_1 imamo :

$$x^2 + y^2 = 1$$

$$(r \cos \varphi)^2 + (r \sin \varphi)^2 = 1$$

$$r^2 (\cos^2 \varphi + \sin^2 \varphi) = 1$$

$$r^2 = 1 \rightarrow r = 1$$

Dakle: $0 \leq r \leq 1$

Ugao ide od x ose do prave $y = \sqrt{3}x$ pa je $0 \leq \varphi \leq \frac{\pi}{3}$

Za oblast D_2 imamo :

$$x^2 + y^2 - 2x = 0$$

$$(r \cos \varphi)^2 + (r \sin \varphi)^2 - 2r \cos \varphi = 0$$

$$r^2 (\cos^2 \varphi + \sin^2 \varphi) = 2r \cos \varphi$$

$$r^2 = 2r \cos \varphi \rightarrow r = 2 \cos \varphi$$

Odavde zaključujemo: $0 \leq r \leq 2 \cos \varphi$

Ugao ide od prave $y = \sqrt{3}x$ pa do y ose pa je $\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{3}$

Imamo dakle:

$$D_1 : \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \varphi \leq \frac{\pi}{3} \end{cases} \quad D_2 : \begin{cases} 0 \leq r \leq 2 \cos \varphi \\ \frac{\pi}{2} \leq \varphi \leq \frac{\pi}{3} \end{cases}$$

Podintegralna funkcija će kad stavimo smene biti:

$$xy = r \cos \varphi \cdot r \sin \varphi = r^2 \sin \varphi \cos \varphi$$

Da rešavamo sada integral:

$$\begin{aligned} \iint_D xy dx dy &= \int_0^{\frac{\pi}{3}} d\varphi \int_0^1 r^2 \sin \varphi \cos \varphi \cdot r dr + \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} d\varphi \int_0^{2 \cos \varphi} r^2 \sin \varphi \cos \varphi \cdot r dr = \\ &= \int_0^{\frac{\pi}{3}} d\varphi \int_0^1 r^3 \sin \varphi \cos \varphi dr + \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} d\varphi \int_0^{2 \cos \varphi} r^3 \sin \varphi \cos \varphi dr \end{aligned}$$

Svaki ćemo posebno , pa ćemo sabrati rešenja:

$$\int_0^{\frac{\pi}{3}} d\varphi \int_0^1 r^3 \sin \varphi \cos \varphi dr = \int_0^{\frac{\pi}{3}} \sin \varphi \cos \varphi d\varphi \int_0^1 r^3 dr = \int_0^{\frac{\pi}{3}} s \sin \varphi \cos \varphi d\varphi \cdot \frac{r^4}{4} \Big|_0^1 = \frac{1}{4} \int_0^{\frac{\pi}{3}} s \sin \varphi \cos \varphi d\varphi$$

$$\int s \sin \varphi \cos \varphi d\varphi = \left| \begin{matrix} \sin \varphi = t \\ \cos \varphi d\varphi = dt \end{matrix} \right| = \int t dt = \frac{t^2}{2} = \frac{\sin^2 \varphi}{2}$$

$$\int_0^{\frac{\pi}{3}} d\varphi \int_0^1 r^3 \sin \varphi \cos \varphi dr = \int_0^{\frac{\pi}{3}} \sin \varphi \cos \varphi d\varphi \int_0^1 r^3 dr = \int_0^{\frac{\pi}{3}} \sin \varphi \cos \varphi d\varphi \cdot \frac{r^4}{4} \bigg|_0^1 = \frac{1}{4} \frac{\sin^2 \varphi}{2} \bigg|_0^{\frac{\pi}{3}} =$$

$$= \frac{1}{8} \left(\sin^2 \frac{\pi}{3} - \sin^2 0 \right) = \frac{1}{8} \left(\frac{\sqrt{3}}{2} \right)^2 = \frac{1}{8} \left(\frac{3}{4} \right) = \boxed{\frac{3}{32}}$$

Sad drugi:

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^3 \sin \varphi \cos \varphi dr = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin \varphi \cos \varphi d\varphi \int_0^{2\cos\varphi} r^3 dr = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin \varphi \cos \varphi d\varphi \cdot \frac{r^4}{4} \bigg|_0^{2\cos\varphi} =$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin \varphi \cos \varphi \cdot \frac{16 \cos^4 \varphi}{4} d\varphi = 4 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin \varphi \cos^5 \varphi d\varphi =$$

Na stranu kao neodređeni pa vratimo granice...

$$\int \sin \varphi \cos^5 \varphi d\varphi = \left| \begin{array}{l} \cos \varphi = t \\ -\sin \varphi d\varphi = dt \end{array} \right| = -\int t^5 dt = -\frac{t^6}{6} = -\frac{\cos^6 \varphi}{6}$$

$$4 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin \varphi \cos^5 \varphi d\varphi = 4 \left(-\frac{\cos^6 \varphi}{6} \right) \bigg|_{\frac{\pi}{3}}^{\frac{\pi}{2}} = -4 \left[\frac{\cos^6 \frac{\pi}{2}}{6} - \frac{\cos^6 \frac{\pi}{3}}{6} \right] = -4 \left[0 - \frac{64}{6} \right] = \boxed{\frac{1}{96}}$$

Konačno je:

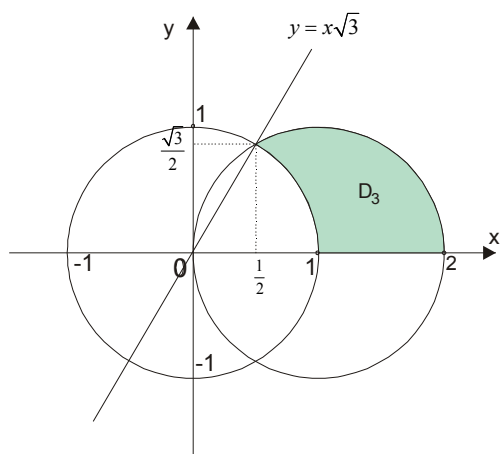
$$\iint_D xy dx dy = \int_0^{\frac{\pi}{3}} d\varphi \int_0^1 r^2 \sin \varphi \cos \varphi \cdot r dr + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2 \sin \varphi \cos \varphi \cdot r dr = \frac{3}{32} + \frac{1}{96} = \boxed{\frac{5}{48}}$$

I ovo bi bilo rešenje našeg zadatka. **ALI!**

PREVIDELI SMO JEDNU STVAR!

Ovde postoji i druga moguća oblast!

Pogledajmo sliku opet.



I ova oblast je ograničena datim kružnicama i x osom u prvom kvadrantu!

Pazite na ovo, zadatak može biti iz dva dela a da vam to profesor ne napomene...

Ovde bi bilo:

$$D_3 : \begin{cases} 1 \leq r \leq 2 \cos \varphi \\ 0 \leq \varphi \leq \frac{\pi}{3} \end{cases}$$

Sličnim rešavanjem kao malopre bi dobili:

$$\iint_D xy dx dy = \int_0^{\frac{\pi}{3}} d\varphi \int_1^{2 \cos \varphi} r^2 \sin \varphi \cos \varphi \cdot r dr = \int_0^{\frac{\pi}{3}} d\varphi \int_1^{2 \cos \varphi} r^3 \sin \varphi \cos \varphi dr = \frac{9}{16}$$

Primer 8.

Izračunati vrednost integrala $\iint_D (y-x) dx dy$ ako je oblast ograničena pravama

$$y = x + 1$$

$$y = x - 3$$

$$y = -\frac{1}{3}x + \frac{7}{3}$$

$$y = -\frac{1}{3}x + 5$$

$$u = y - x$$

stavljajući da je:

$$v = y + \frac{1}{3}x$$

Rešenje:

Da se podsetimo:

Ako se sa $x=x(u,v)$ i $y=y(u,v)$, gde su ovo neprekidne i diferencijabilne funkcije, realizuje jednoznačno preslikavanje ograničene i zatvorene oblasti D u ravni xOy na oblast D' u ravni uOv i ako je:

$$J = \frac{D(x,y)}{D(u,v)} \neq 0$$

Onda važi formula:

$$\iint_D z(x,y) dx dy = \iint_{D'} z[x(u,v), y(u,v)] |J| du dv$$

Nemamo mnogo da mozgamo, jer su nam dati u i v .

Ovde nam je prvi poso da izrazimo x i y :

$$u = y - x$$

$$v = y + \frac{1}{3}x \dots \dots \dots / *3$$

$$u = y - x$$

$$3v = 3y + x$$

$$u + 3v = 4y \rightarrow \boxed{y = \frac{1}{4}u + \frac{3}{4}v}$$

$$u = y - x \dots \dots \dots * (-3)$$

$$3v = 3y + x$$

$$-3u = -3y + 3x$$

$$3v = 3y + x$$

$$-3u + 3v = 4x \rightarrow \boxed{x = -\frac{3}{4}u + \frac{3}{4}v}$$

Sad tražimo Jakobijan koji mora biti različit od nule:

$$\frac{D(x,y)}{D(u,v)} = \begin{vmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{4} \end{vmatrix} = -\frac{9}{16} - \frac{3}{16} = -\frac{12}{16} = -\frac{3}{4}$$

Naravno, mi uzimamo apsolutnu vrednost (po formuli $\iint_D z(x,y) dx dy = \iint_{D'} z[x(u,v), y(u,v)] |J| du dv$) to

jest, kod nas je $|J| = \frac{3}{4}$

Kako odrediti granice za u i v ?

Posmatrajmo granice po x i y :

$$\left. \begin{array}{l} y = x + 1 \\ y = x - 3 \end{array} \right\} \rightarrow \left. \begin{array}{l} y - x = 1 \\ y - x = -3 \end{array} \right\} \rightarrow \left. \begin{array}{l} u = 1 \\ u = -3 \end{array} \right\} \rightarrow \boxed{-3 \leq u \leq 1}$$

$$\left. \begin{array}{l} y = -\frac{1}{3}x + \frac{7}{3} \\ y = -\frac{1}{3}x + 5 \end{array} \right\} \rightarrow \left. \begin{array}{l} y + \frac{1}{3}x = \frac{7}{3} \\ y + \frac{1}{3}x = 5 \end{array} \right\} \rightarrow \left. \begin{array}{l} v = \frac{7}{3} \\ v = 5 \end{array} \right\} \rightarrow \boxed{\frac{7}{3} \leq v \leq 5}$$

U datom integralu moramo zameniti x i y iz onog što smo izrazili:

$$\begin{aligned} \iint_D (y-x) dx dy &= \iint_{D'} \left(\left(\frac{1}{4}u + \frac{3}{4}v \right) - \left(-\frac{3}{4}u + \frac{3}{4}v \right) \right) \cdot \frac{3}{4} du dv = \iint_{D'} \left(\frac{1}{4}u + \frac{3}{4}v + \frac{3}{4}u - \frac{3}{4}v \right) \cdot \frac{3}{4} du dv = \\ &= \iint_{D'} u \cdot \frac{3}{4} du dv = \int_{\frac{7}{3}}^5 \left(\frac{3}{4} \int_{-3}^1 u du \right) dv = \frac{3}{4} \int_{\frac{7}{3}}^5 \left(\frac{u^2}{2} \Big|_{-3}^1 \right) dv = \frac{3}{4} \int_{\frac{7}{3}}^5 \left(\frac{1^2}{2} - \frac{(-3)^2}{2} \right) dv = \frac{3}{4} \int_{\frac{7}{3}}^5 (-4) dv = \\ &= -3 \int_{\frac{7}{3}}^5 dv = -3v \Big|_{\frac{7}{3}}^5 = -3 \left(5 - \frac{7}{3} \right) = -3 \cdot \frac{8}{3} = \boxed{-8} \end{aligned}$$

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