

INTEGRALI ZADACI (V-DEO)

Integrali nekih funkcija koje sadrže kvadratni trinom $ax^2 + bx + c$

Najpre ćemo proučiti integrale oblika:

$$I_1 = \int \frac{dx}{ax^2 + bx + c} \quad \text{i} \quad I_3 = \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

Kod njih se kvadratni trinom $ax^2 + bx + c$ svede na kanonični oblik pomoću formule:

$$ax^2 + bx + c = a \left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a}$$

naravno, možemo koristiti i dopunu do punog kvadrata, ko ne voli da pamti formulu.

Zatim uzimamo smenu: $\left| \begin{array}{l} x + \frac{b}{2a} = t \\ dx = dt \end{array} \right|$, i dobijemo neki od tabličnih integrala.

$$I_1 = \int \frac{dx}{ax^2 + bx + c} \quad \text{se može svesti na} \quad \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C, \quad \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C$$

ili $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$

$$I_3 = \int \frac{dx}{\sqrt{ax^2 + bx + c}} \quad \text{se svodi najčešće na} \quad \int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C \quad \text{ili} \quad \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C$$

primer 1. $\int \frac{dx}{x^2 - 6x + 13} = ?$

$x^2 - 6x + 13 =$ Ovde je $a=1$, $b=-6$, $c=13$ pa to zamenimo u formulu $a \left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a}$, dakle:

$$a \left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a} = 1 \left(x + \frac{-6}{2 \cdot 1} \right)^2 + \frac{4 \cdot 1 \cdot 13 - (-6)^2}{4 \cdot 1} = (x-3)^2 + \frac{52-36}{4} = \boxed{(x-3)^2 + 4}$$

Lakše je naravno izvršiti dopunu do punog kvadrata, znate ono dodamo i oduzmemo onaj uz x podeljen sa 2 pa to na

kvadrat. $\left(\frac{\text{broj uz } x}{2} \right)^2$

$$x^2 - 6x + 13 = \underline{x^2 - 6x + 9} - 9 + 13 = \boxed{(x-3)^2 + 4} \quad \text{Uz } x \text{ je } 6, \text{ pa dodajemo i oduzimamo } \left(\frac{6}{2} \right)^2 = 9$$

Vraćamo se u integral:

$$\int \frac{dx}{x^2 - 6x + 13} = \int \frac{dx}{(x-3)^2 + 4} = \left| \begin{matrix} x-3=t \\ dx=dt \end{matrix} \right| = \int \frac{dt}{t^2 + 2^2} \quad (\text{ovo je } \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C \text{ iz tabele}) =$$

$$\frac{1}{2} \operatorname{arctg} \frac{t}{2} + C = (\text{vratimo smenu}) = \boxed{\frac{1}{2} \operatorname{arctg} \frac{x-3}{2} + C}$$

primer 2. $\int \frac{dx}{\sqrt{2x^2 - 6x + 5}} = ?$

$$2x^2 - 6x + 5 = 2(x^2 - 3x + \frac{5}{2}) = 2(x^2 - 3x + \frac{9}{4} - \frac{9}{4} + \frac{5}{2}) = 2[(x - \frac{3}{2})^2 + \frac{1}{4}]$$

$$\int \frac{dx}{\sqrt{2x^2 - 6x + 5}} = \int \frac{dx}{\sqrt{2[(x - \frac{3}{2})^2 + \frac{1}{4}]}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(x - \frac{3}{2})^2 + \frac{1}{4}}} = \left| \begin{matrix} x - \frac{3}{2} = t \\ dx = dt \end{matrix} \right| = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{t^2 + (\frac{1}{2})^2}}$$

Upotrebimo iz tablice $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln |x + \sqrt{x^2 \pm a^2}| + C$

$$\frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{t^2 + (\frac{1}{2})^2}} = \frac{1}{\sqrt{2}} \ln \left| t + \sqrt{t^2 + (\frac{1}{2})^2} \right| + C = \text{vratimo smenu} = \boxed{\frac{1}{\sqrt{2}} \ln \left| x - \frac{3}{2} + \sqrt{2x^2 - 6x + 5} \right| + C}$$

Kad znamo ova dva tipa integrala, možemo naučiti i :

$$\boxed{I_2 = \int \frac{Ax + B}{ax^2 + bx + c} dx} \quad \text{i} \quad \boxed{I_4 = \int \frac{Ax + B}{\sqrt{ax^2 + bx + c}} dx}$$

Oni se radom svedu na prethodna dva integrala:

$$I_2 = \int \frac{Ax + B}{ax^2 + bx + c} dx \quad \text{se svede na integral} \quad I_1 = \int \frac{dx}{ax^2 + bx + c}, \text{ dok se}$$

$$I_4 = \int \frac{Ax + B}{\sqrt{ax^2 + bx + c}} dx \quad \text{svede na integral} \quad I_3 = \int \frac{dx}{\sqrt{ax^2 + bx + c}}.$$

Postoje gotove formule u kojima treba samo da uporedite i nadjete vrednosti za A, B, a, b i c .

Pazite: njih smete koristiti samo ako to odobrava vaš profesor! Mi ćemo vam pokazati i ceo postupak u slučaju da ne smete da koristite formule...

Formulice su:

$$I_2 = \frac{A}{2a} \ln |ax^2 + bx + c| + \left(B - \frac{Ab}{2a}\right) I_1 + C \quad \text{i} \quad I_4 = \frac{A}{a} \sqrt{ax^2 + bx + c} + \left(B - \frac{A \cdot b}{2a}\right) I_3 + C$$

primer 3. $\int \frac{x+1}{x^2+x+1} dx = ?$

Ovo je očigledno integral tipa $I_2 = \int \frac{Ax+B}{ax^2+bx+c} dx$

Upoređivanjem dobijamo da je : $A=1, B=1, a=1, b=1, c=1$

$$I_2 = \frac{A}{2a} \ln |ax^2 + bx + c| + \left(B - \frac{Ab}{2a}\right) I_1 + C$$

$$A=1, B=1, a=1, b=1, c=1$$

$$I_2 = \frac{1}{2 \cdot 1} \ln |1x^2 + 1x + 1| + \left(1 - \frac{1 \cdot 1}{2 \cdot 1}\right) I_1 + C = \left[\frac{1}{2} \ln |x^2 + x + 1| + \frac{1}{2} I_1 + C \right]$$

Sad imamo poso da rešimo integral tipa $I_1 = \int \frac{1}{x^2+x+1} dx$ i da njegovo rešenje posle vratimo u formulu.

$$I_1 = \int \frac{1}{x^2+x+1} dx = ?$$

$$x^2+x+1 = x^2+x+\frac{1}{4}-\frac{1}{4}+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\begin{aligned} I_1 &= \int \frac{1}{x^2+x+1} dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} dx = \left[\begin{matrix} x+\frac{1}{2} = t \\ dx = dt \end{matrix} \right] = \int \frac{1}{t^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx = \frac{1}{\frac{\sqrt{3}}{2}} \operatorname{arctg} \frac{t}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2(x+\frac{1}{2})}{\sqrt{3}} = \\ &= \left[\frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} \right] \end{aligned}$$

Vratimo se u formulu:

$$\begin{aligned} \int \frac{x+1}{x^2+x+1} dx &= \frac{1}{2} \ln |x^2+x+1| + \frac{1}{2} I_1 + C = \frac{1}{2} \ln |x^2+x+1| + \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} + C \\ &= \left[\frac{1}{2} \ln |x^2+x+1| + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} + C \right] \end{aligned}$$

Kako bi ovaj integral rešavali da nismo smeli koristiti formulu?

$$\int \frac{x+1}{x^2+x+1} dx = ?$$

Ideja je da se izraz u brojiocu $Ax+B$ napravi da bude izvod izraza u imeniocu ax^2+bx+c .

To možete uraditi tako što izvučete ispred integrala $\frac{A}{2a}$.

U našem primeru imamo x^2+x+1 u imeniocu, njegov izvod je $(x^2+x+1)' = 2x+1$, što znači da u brojiocu treba da

napravimo $2x+1$, odnosno da izvučemo $\frac{A}{2a} = \frac{1}{2}$ ispred integrala!

$$\int \frac{x+1}{x^2+x+1} dx = \frac{1}{2} \int \frac{2x+2}{x^2+x+1} dx = \frac{1}{2} \int \frac{2x+1+1}{x^2+x+1} dx = \frac{1}{2} \left(\int \frac{2x+1}{x^2+x+1} dx + \int \frac{1}{x^2+x+1} dx \right)$$

Sad se problem sveo na rešavanje dva integrala, gde prvi uvek radimo smenom, a drugi je tipa I_1 .

$$\int \frac{2x+1}{x^2+x+1} dx = \left| \begin{array}{l} x^2+x+1=t \\ (2x+1)dx=dt \end{array} \right| = \int \frac{1}{t} dt = \ln|t| = \ln|x^2+x+1|$$

Ovaj drugi smo već rešavali:

$$\begin{aligned} \int \frac{1}{x^2+x+1} dx &= \int \frac{1}{x^2+x+1} dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} dx = \left| \begin{array}{l} x+\frac{1}{2}=t \\ dx=dt \end{array} \right| = \int \frac{1}{t^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx = \frac{1}{\frac{\sqrt{3}}{2}} \operatorname{arctg} \frac{t}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2(x+\frac{1}{2})}{\sqrt{3}} = \\ &= \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} \end{aligned}$$

Vratimo se na zadatak :

$$\begin{aligned} \int \frac{x+1}{x^2+x+1} dx &= \frac{1}{2} \left(\int \frac{2x+1}{x^2+x+1} dx + \int \frac{1}{x^2+x+1} dx \right) = \\ &= \frac{1}{2} (\ln|x^2+x+1| + \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}}) + C \\ &= \boxed{\frac{1}{2} \ln|x^2+x+1| + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} + C} \end{aligned}$$

primer 4. $\int \frac{5x+3}{\sqrt{x^2+4x+10}} dx = ?$

I način (uz pomoć formule)

$$I_4 = \frac{A}{a} \sqrt{ax^2 + bx + c} + \left(B - \frac{A \cdot b}{2a} \right) I_3$$

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}} dx = ?$$

$$A = 5, B = 3, a = 1, b = 4, c = 10$$

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}} dx = \frac{5}{1} \sqrt{x^2+4x+10} + \left(3 - \frac{5 \cdot 4}{2 \cdot 1} \right) \int \frac{1}{\sqrt{x^2+4x+10}} dx$$

$$= 5 \cdot \sqrt{x^2+4x+10} + (-7) \int \frac{1}{\sqrt{x^2+4x+10}} dx$$

$$= 5 \cdot \sqrt{x^2+4x+10} - 7 \cdot \int \frac{1}{\sqrt{x^2+4x+10}} dx$$

Da rešimo ovaj integral posebno, pa ćemo vratiti njegovo rešenje...

$$\int \frac{1}{\sqrt{x^2+4x+10}} dx =$$

$$x^2 + 4x + 10 = x^2 + 4x + 4 - 4 + 10 = (x+2)^2 + 6$$

$$\int \frac{1}{\sqrt{x^2+4x+10}} dx = \int \frac{1}{\sqrt{(x+2)^2+6}} dx = \left| \begin{matrix} x+2=t \\ dx=dt \end{matrix} \right| = \int \frac{1}{\sqrt{t^2+6}} dx = \text{koristimo: } \boxed{\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C}$$

$$= \ln \left| t + \sqrt{t^2+6} \right| = \ln \left| x+2 + \sqrt{x^2+4x+10} \right|$$

$$\begin{aligned} \int \frac{5x+3}{\sqrt{x^2+4x+10}} dx &= 5 \cdot \int \frac{x+2}{\sqrt{x^2+4x+10}} dx - 7 \cdot \int \frac{1}{\sqrt{x^2+4x+10}} dx \\ &= \boxed{5 \cdot \sqrt{x^2+4x+10} - 7 \cdot \ln \left| x+2 + \sqrt{x^2+4x+10} \right| + C} \end{aligned}$$

II način (direktno, bez upotrebe formule)

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}} dx = ?$$

Kako je izvod $(x^2+4x+10)' = 2x+4 = 2(x+2)$ u brojiocu mora biti napravljeno to.

$$\begin{aligned}
\int \frac{5x+3}{\sqrt{x^2+4x+10}} dx &= \int \frac{5(x+\frac{3}{5})}{\sqrt{x^2+4x+10}} dx = 5 \cdot \int \frac{x+2-2+\frac{3}{5}}{\sqrt{x^2+4x+10}} dx = \\
&= 5 \cdot \left(\int \frac{x+2}{\sqrt{x^2+4x+10}} dx + \int \frac{-2+\frac{3}{5}}{\sqrt{x^2+4x+10}} dx \right) \\
&= 5 \cdot \left(\int \frac{x+2}{\sqrt{x^2+4x+10}} dx + \int \frac{-\frac{7}{5}}{\sqrt{x^2+4x+10}} dx \right) \\
&= 5 \cdot \int \frac{x+2}{\sqrt{x^2+4x+10}} dx - 7 \cdot \int \frac{1}{\sqrt{x^2+4x+10}} dx
\end{aligned}$$

Sad radimo ova dva integrala (drugi smo već rešavali kod prvog načina).

$$\int \frac{x+2}{\sqrt{x^2+4x+10}} dx = \left. \begin{array}{l} x^2+4x+10=t^2 \\ (2x+4)dx=2tdt \\ \cancel{2}(x+2)dx=\cancel{2}tdt \\ (x+2)dx=tdt \end{array} \right| = \int \frac{\cancel{2}tdt}{\cancel{2}} = \int dt = t = \sqrt{x^2+4x+10}$$

$$\begin{aligned}
&\int \frac{1}{\sqrt{x^2+4x+10}} dx = \\
&x^2+4x+10 = x^2+4x+4-4+10 = (x+2)^2+6 \\
&\int \frac{1}{\sqrt{x^2+4x+10}} dx = \int \frac{1}{\sqrt{(x+2)^2+6}} dx = \left| \begin{array}{l} x+2=t \\ dx=dt \end{array} \right| = \int \frac{1}{\sqrt{t^2+6}} dx = \text{koristimo: } \boxed{\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C} \\
&= \ln \left| t + \sqrt{t^2+6} \right| = \ln \left| x+2 + \sqrt{x^2+4x+10} \right|
\end{aligned}$$

Vratimo se u zadatak:

$$\begin{aligned}
\int \frac{5x+3}{\sqrt{x^2+4x+10}} dx &= 5 \cdot \int \frac{x+2}{\sqrt{x^2+4x+10}} dx - 7 \cdot \int \frac{1}{\sqrt{x^2+4x+10}} dx \\
&= \boxed{5 \cdot \sqrt{x^2+4x+10} - 7 \cdot \ln \left| x+2 + \sqrt{x^2+4x+10} \right| + C}
\end{aligned}$$

primer 5. $\int \frac{2x+7}{x^2+x-2} dx = ?$

Ovaj primer vam navodimo jer trebate voditi računa o polinomu u imeniocu!

Rekli bi da je ovo integral tipa $I_2 = \int \frac{Ax+B}{ax^2+bx+c} dx$ i radili bi:

$$\int \frac{2x+7}{x^2+x-2} dx = \int \frac{2x+1+6}{x^2+x-2} dx = \int \frac{2x+1}{x^2+x-2} dx + \int \frac{6}{x^2+x-2} dx =$$

Rešimo ova dva integrala posebno, pa ćemo zameniti njihova rešenja...

$$\int \frac{2x+1}{x^2+x-2} dx = \left| \begin{array}{l} x^2+x-2=t \\ (2x+1)dx=dt \end{array} \right| = \int \frac{dt}{t} = \ln|t| = \ln|x^2+x-2|$$

$$\int \frac{6}{x^2+x-2} dx = 6 \int \frac{1}{x^2+x-2} dx =$$

$$x^2+x-2 = x^2+x+\frac{1}{4}-\frac{1}{4}-2 = \left(x+\frac{1}{2}\right)^2 - \frac{9}{4}$$

$$6 \int \frac{1}{x^2+x-2} dx = 6 \int \frac{1}{\left(x+\frac{1}{2}\right)^2 - \frac{9}{4}} dx = \left| \begin{array}{l} x+\frac{1}{2}=t \\ dx=dt \end{array} \right| = 6 \int \frac{1}{t^2 - \left(\frac{3}{2}\right)^2} dx = \text{koristimo: } \boxed{\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C}$$

$$= 6 \cdot \frac{1}{2 \cdot \frac{3}{2}} \ln \left| \frac{t - \frac{3}{2}}{t + \frac{3}{2}} \right| = 2 \ln \left| \frac{x + \frac{1}{2} - \frac{3}{2}}{x + \frac{1}{2} + \frac{3}{2}} \right| = 2 \ln \left| \frac{x-1}{x+2} \right|$$

vratimo rešenja:

$$\int \frac{2x+7}{x^2+x-2} dx = \int \frac{2x+1}{x^2+x-2} dx + \int \frac{6}{x^2+x-2} dx =$$

$$= \ln|x^2+x-2| + 2 \ln \left| \frac{x-1}{x+2} \right| + C$$

$$= \ln|(x-1)(x+2)| + \ln \left| \frac{x-1}{x+2} \right|^2 + C$$

$$= \ln \left| (x-1) \cancel{(x+2)} \cdot \frac{(x-1)^2}{(x+2)^{\cancel{2}}} \right| + C$$

$$\boxed{= \ln \left| \frac{(x-1)^3}{(x+2)} \right| + C}$$

Nije bilo lako rešiti ga, priznaćete...

Nismo razmišljali jednu drugu stvar: Da li je ovaj zadatak mogao da se uradi kao integracija racionalne funkcije?

Proverimo da li polinom u imeniocu može da se rastavi na činioce...

$$x^2 + x - 2 = 0 \rightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow x_1 = 1, x_2 = -2 \quad \text{MOŽE!}$$

Lakše je raditi (bar nama):

$$\int \frac{2x+7}{x^2+x-2} dx = ?$$
$$\frac{2x+7}{x^2+x-2} = \frac{2x+7}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \dots\dots\dots / \square (x-1)(x+2)$$

$$2x+7 = A(x+2) + B(x-1)$$

$$2x+7 = Ax + 2A + Bx - B$$

$$2x+7 = x(A+B) + 2A - B$$

upoređujemo

$$A+B=2$$

$$2A-B=7$$

$$3A=9 \rightarrow \boxed{A=3} \rightarrow \boxed{B=-1}$$

$$\frac{2x+7}{(x-1)(x+2)} = \frac{3}{x-1} + \frac{-1}{x+2} = \frac{3}{x-1} - \frac{1}{x+2}$$

$$\int \frac{2x+7}{(x-1)(x+2)} dx = \int \frac{3}{x-1} dx - \int \frac{1}{x+2} dx = 3 \ln|x-1| - \ln|x+2| + C$$
$$= \ln|x-1|^3 - \ln|x+2| + C$$

$$\boxed{= \ln \left| \frac{(x-1)^3}{x+2} \right| + C}$$

Naš savet je dakle da proverite da li je kvadratna jednačina u imeniocu rešiva i da ako jeste radite integral kao integraciju racionalne funkcije.

Ako kvadratna nije rešiva, radite ga kao integral tipa $I_2 = \int \frac{Ax+B}{ax^2+bx+c} dx$.

Videli ste da su ispala ista rešenja.

Uostalom, odlučite sami, šta vama više odgovara ili kako pak komanduje profesor...

Sledeći tip integrala je

$$\int \frac{dx}{(mx+n)\sqrt{ax^2+bx+c}}$$

Ovi integrali se smenom:
$$\begin{cases} mx+n = \frac{1}{t} \\ m dx = -\frac{1}{t^2} dt \\ dx = -\frac{1}{m \cdot t^2} dt \end{cases}, \text{ svedu na integral tipa } I_3 = \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

primer 6.
$$\int \frac{dx}{x \cdot \sqrt{x^2 - 4x + 1}} = ?$$

Najpre uzimamo smenu $x = \frac{1}{t}$ kojom svodimo dati integral na tip I_3 .

$$\begin{aligned} \int \frac{dx}{x \cdot \sqrt{x^2 - 4x + 1}} &= \left| \begin{array}{l} x = \frac{1}{t} \\ dx = -\frac{1}{t^2} dt \end{array} \right| = \int \frac{-\frac{1}{t^2} dt}{\frac{1}{t} \cdot \sqrt{\left(\frac{1}{t}\right)^2 - 4 \cdot \frac{1}{t} + 1}} = \int \frac{-\frac{1}{t^2} dt}{\frac{1}{t} \cdot \sqrt{\frac{1}{t^2} - \frac{4}{t} + 1}} = \int \frac{-\frac{1}{t} dt}{\sqrt{\frac{1 - 4t + t^2}{t^2}}} = \\ &= \int \frac{-\frac{1}{t} dt}{\frac{1}{t} \sqrt{t^2 - 4t + 1}} = \int \frac{-dt}{\sqrt{t^2 - 4t + 1}} = -\int \frac{dt}{\sqrt{t^2 - 4t + 1}} \end{aligned}$$

$$t^2 - 4t + 1 = t^2 - 4t + 4 - 4 + 1 = (t-2)^2 - 3$$

$$\begin{aligned} -\int \frac{dt}{\sqrt{t^2 - 4t + 1}} &= -\int \frac{dt}{\sqrt{(t-2)^2 - 3}} = \left| \begin{array}{l} t-2 = z \\ dt = dz \end{array} \right| = -\int \frac{dz}{\sqrt{z^2 - 3}} = \text{koristimo: } \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C, \text{ pa je} \\ &= -\ln \left| z + \sqrt{z^2 - 3} \right| = \text{vratimo smenu} = -\ln \left| t-2 + \sqrt{t^2 - 4t + 1} \right| + C \end{aligned}$$

Moramo da vratimo i prvu smenu:

$$= -\ln \left| t-2 + \sqrt{t^2 - 4t + 1} \right| + C = \boxed{-\ln \left| \frac{1}{x} - 2 + \sqrt{\left(\frac{1}{x}\right)^2 - 4 \cdot \frac{1}{x} + 1} \right| + C}$$

Metoda neodređenih koeficijenata (metoda Ostrogradskog)

Ovom metodom se rešavaju integrali tipa $\int \frac{P_n(x)}{\sqrt{ax^2 + bx + c}} dx$ gde je u brojiocu podintegralne funkcije imamo polinom n-tog stepena.

Postupak rada je sledeći:

- postavimo jednačinu

$$\int \frac{P_n(x)}{\sqrt{ax^2 + bx + c}} dx = Q_{n-1}(x) \cdot \sqrt{ax^2 + bx + c} + \lambda \cdot \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

Ovde je $Q_{n-1}(x)$ polinom (n-1) vog stepena sa neodređenim koeficijentima.

- ovu jednačinu diferenciramo
- zatim sve pomnožimo sa $\sqrt{ax^2 + bx + c}$
- sa obe strane dobijamo polinome reda n , pa neodređene koeficijente odredjujemo izjednačavanjem koeficijenata uz iste stepene x-a.

Kako je polinom $P_n(x)$ u zadacima najčešće drugog stepena početna jednačina će biti:

$$\int \frac{mx^2 + px + r}{\sqrt{ax^2 + bx + c}} dx = (Ax+B) \sqrt{ax^2 + bx + c} + \lambda \cdot \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

Ali, najbolje da to vidimo na primeru:

primer 7. $\int \frac{2x^2 + 3x}{\sqrt{x^2 + 2x + 2}} dx = ?$

Postavimo jednačinu:

$$\int \frac{2x^2 + 3x}{\sqrt{x^2 + 2x + 2}} dx = (Ax + B) \cdot \sqrt{x^2 + 2x + 2} + \lambda \cdot \int \frac{dx}{\sqrt{x^2 + 2x + 2}} \rightarrow \text{diferenciramo}$$

$$\frac{2x^2 + 3x}{\sqrt{x^2 + 2x + 2}} = (Ax + B) \cdot \sqrt{x^2 + 2x + 2} + (\sqrt{x^2 + 2x + 2})' \cdot (Ax + B) + \frac{\lambda}{\sqrt{x^2 + 2x + 2}}$$

$$\frac{2x^2 + 3x}{\sqrt{x^2 + 2x + 2}} = A \cdot \sqrt{x^2 + 2x + 2} + \frac{1}{2\sqrt{x^2 + 2x + 2}} (x^2 + 2x + 2)' \cdot (Ax + B) + \frac{\lambda}{\sqrt{x^2 + 2x + 2}}$$

$$\frac{2x^2 + 3x}{\sqrt{x^2 + 2x + 2}} = A \cdot \sqrt{x^2 + 2x + 2} + \frac{1}{2\sqrt{x^2 + 2x + 2}} (2x + 2) \cdot (Ax + B) + \frac{\lambda}{\sqrt{x^2 + 2x + 2}}$$

$$\frac{2x^2 + 3x}{\sqrt{x^2 + 2x + 2}} = A \cdot \sqrt{x^2 + 2x + 2} + \frac{1}{\cancel{2}\sqrt{x^2 + 2x + 2}} \cancel{2}(x + 1) \cdot (Ax + B) + \frac{\lambda}{\sqrt{x^2 + 2x + 2}}$$

$$\frac{2x^2 + 3x}{\sqrt{x^2 + 2x + 2}} = A \cdot \sqrt{x^2 + 2x + 2} + \frac{(x + 1) \cdot (Ax + B)}{\sqrt{x^2 + 2x + 2}} + \frac{\lambda}{\sqrt{x^2 + 2x + 2}} \rightarrow \dots\dots\dots / \sqrt{x^2 + 2x + 2}$$

$$2x^2 + 3x = A(x^2 + 2x + 2) + (x + 1) \cdot (Ax + B) + \lambda$$

Sada upoređujemo koeficijente:

$$2x^2 + 3x = A(x^2 + 2x + 2) + (x + 1) \cdot (Ax + B) + \lambda$$

$$2x^2 + 3x = \underline{Ax^2} + \underline{2Ax} + \underline{2A} + \underline{Ax^2} + \underline{Bx} + \underline{Ax} + \underline{B} + \lambda$$

$$2x^2 + 3x = 2Ax^2 + x(3A + B) + 2A + B + \lambda \rightarrow \text{upoređujemo}$$

$$2A = 2 \rightarrow \boxed{A = 1}$$

$$3A + B = 3$$

$$\underline{2A + B + \lambda = 0}$$

$$3A + B = 3 \rightarrow 3 \cdot 1 + B = 3 \rightarrow \boxed{B = 0}$$

$$2A + B + \lambda = 0 \rightarrow 2 + 0 + \lambda = 0 \rightarrow \boxed{\lambda = -2}$$

Vratimo se u početnu jednačinu:

$$\int \frac{2x^2 + 3x}{\sqrt{x^2 + 2x + 2}} dx = (Ax + B) \cdot \sqrt{x^2 + 2x + 2} + \lambda \cdot \int \frac{dx}{\sqrt{x^2 + 2x + 2}}$$

$$= (1x + 0) \cdot \sqrt{x^2 + 2x + 2} - 2 \cdot \int \frac{dx}{\sqrt{x^2 + 2x + 2}}$$

$$= x \cdot \sqrt{x^2 + 2x + 2} - 2 \cdot \int \frac{dx}{\sqrt{x^2 + 2x + 2}}$$

Da rešimo posebno ovaj integral...

$$\int \frac{dx}{\sqrt{x^2+2x+2}} = \int \frac{dx}{\sqrt{x^2+2x+1+1}} = \int \frac{dx}{\sqrt{(x+1)^2+1}} = \left| \begin{matrix} x+1=t \\ dx=dt \end{matrix} \right| = \int \frac{dx}{\sqrt{t^2+1}} =$$

$$\text{upotrebimo: } \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C$$

$$= \ln \left| t + \sqrt{t^2+1} \right| + C = \boxed{\ln \left| x+1 + \sqrt{x^2+2x+2} \right| + C}$$

Konačno, rešenje će biti:

$$\begin{aligned} \int \frac{2x^2+3x}{\sqrt{x^2+2x+2}} dx &= x \cdot \sqrt{x^2+2x+2} - 2 \cdot \int \frac{dx}{\sqrt{x^2+2x+2}} \\ &= \boxed{x \cdot \sqrt{x^2+2x+2} - 2 \cdot \ln \left| x+1 + \sqrt{x^2+2x+2} \right| + C} \end{aligned}$$

primer 8.

$$\int \sqrt{a^2 - x^2} dx = ?$$

Ako se sećate, ovaj integral smo rešavali u fajlu parcijalna integracija. Tada smo rekli da on može da se rešava na više načina. Evo kako bi išlo rešavanje metodom Ostrogradskog.

Naravno, opet racionalizacijom malo prepravimo podintegralnu funkciju...

$$\begin{aligned} \frac{\sqrt{a^2-x^2}}{1} \cdot \frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} &= \frac{a^2-x^2}{\sqrt{a^2-x^2}} \\ \int \sqrt{a^2-x^2} dx &= \int \frac{a^2-x^2}{\sqrt{a^2-x^2}} dx \end{aligned}$$

Sada je ovo oblik koji nam treba...

$$\int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx = (Ax + B) \cdot \sqrt{a^2 - x^2} + \lambda \cdot \int \frac{dx}{\sqrt{a^2 - x^2}} \rightarrow \text{diferenciramo}$$

$$\frac{a^2 - x^2}{\sqrt{a^2 - x^2}} = A\sqrt{a^2 - x^2} + \frac{-\cancel{2}x}{\cancel{2}\sqrt{a^2 - x^2}}(Ax + B) + \frac{\lambda}{\sqrt{a^2 - x^2}}$$

$$\frac{a^2 - x^2}{\sqrt{a^2 - x^2}} = A\sqrt{a^2 - x^2} + \frac{-x}{\sqrt{a^2 - x^2}}(Ax + B) + \frac{\lambda}{\sqrt{a^2 - x^2}} \dots \dots \dots / \sqrt{a^2 - x^2}$$

$$a^2 - x^2 = A(a^2 - x^2) - x(Ax + B) + \lambda$$

$$a^2 - x^2 = Aa^2 - Ax^2 - Ax^2 - Bx + \lambda$$

$$a^2 - x^2 = -2Ax^2 - Bx + Aa^2 + \lambda$$

upoređujemo

$$-2A = -1$$

$$-B = 0 \rightarrow \boxed{B = 0}$$

$$\underline{Aa^2 + \lambda = a^2}$$

Rešavamo ovaj sistemčić

$$\boxed{A = \frac{1}{2}} \rightarrow \frac{1}{2}a^2 + \lambda = a^2 \rightarrow \boxed{\lambda = \frac{1}{2}a^2}$$

Vratimo se u postavku...

$$\int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx = (Ax + B) \cdot \sqrt{a^2 - x^2} + \lambda \cdot \int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$\int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx = \frac{1}{2}x \cdot \sqrt{a^2 - x^2} + \frac{1}{2}a^2 \cdot \int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$\boxed{\int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} dx = \frac{1}{2}x \cdot \sqrt{a^2 - x^2} + \frac{1}{2}a^2 \cdot \arcsin \frac{x}{a} + C}$$